PACIFIC JOURNAL OF MATHEMATICS
Vol. 78, No. 1, 1978

INTEGER MULTIPLES OF PERIODIC CONTINUED FRACTIONS

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This paper contains much simpler proofs of the results of Henri Cohen (Acta Arithmetica 26 (1974-75), 129-148) on the period length of the continued fraction for $N\alpha$, where N is a positive integer and α is a quadratic irrational.

1. Introduction. We let $[a_0, a_1, \cdots]$ denote the simple continued fraction whose partial quotients are the integers a_i ($a_i > 0$ for $i > 0$). If α is a quadratic irrational, so that α has a periodic continued fraction, then we put

$$\alpha = [b_0, b_1, \cdots, b_m, \overline{a_1, \cdots, a_n}] ,$$

where $b_0, b_1, \cdots, b_m$ is the nonperiodic part of the continued fraction and $a_1, \cdots, a_n$ is the period. We let $P(\alpha) = n$ denote the length of the period of the expansion of α.

H. Cohen [2] defined the functions

$$S(N, n) = \sup_{P(\alpha)=n} P(N\alpha)$$

for each pair of integers $N > 1$, $n \geqq 1$. The fact that $S(N, n)$ is always finite was already known (see Schinzel [4]).

Let A denote the set of all real quadratic irrationals. Cohen defined the function

$$R(N) = \sup_{n\geqq 1} (S(N, n)/n) = \sup_{\alpha\in A} (P(N\alpha)/P(\alpha))$$

for each integer $N > 1$, and proved that $R(N)$ is always finite. The paper of Cohen [2] is devoted to proving various results about $S(N, n)$ and $R(N)$. In particular, Cohen [2, pp. 141-147] obtained the exact value of $R(N)$ for infinitely many N and gave a conjecture for the value of $R(N)$ in all the remaining cases.

Cohen made use of an algorithm given by Mendès France [3] for computing the continued fraction expansion of $N\alpha$ from the expansion of α, where α is any real number. Cohen [2, §§3 and 4, pp. 132-137] devotes considerable space to showing that if one wants to use the algorithm of Mendès France [3] in order to study $P(N\alpha)$ for quadratic irrationals α, then one needs various facts about 2 by 2 matrices with integer entries taken mod N.

It turns out that the algorithm of Mendès France [3] was already given by A. Châtelet [1] in a different but equivalent form. The