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FOCAL SETS OF SUBMANIFOLDS

THOMAS E. CECIL AND PATRICK J. RYAN

This is a study of the manifold structure of the focal set of an immersed submanifold in a real space form $\tilde{M}$. A typical result is the following:

THEOREM. *Let M be an orientable (immersed) hypersurface in $\tilde{M}$ which is complete with respect to the induced metric. Let λ be a differentiable principal curvature of constant multiplicity $\nu > 1$ on M. Then the focal map f_λ factors through an immersion of the $(n-\nu)$-dimensional manifold M_λ into $\tilde{M}$. In this way, $f_\lambda(M)$ is an immersed submanifold of $\tilde{M}$.*

We explain the notation used in the theorem. Under the hypotheses, the principal vectors corresponding to λ form a smooth ν-dimensional distribution T_λ on M whose leaves are umbilic submanifolds of $\tilde{M}$. On each leaf, λ is constant. f_λ is the map from part of M onto the set of focal points arising from the principal curvature λ. M_λ is the space of leaves of T_λ which intersect the domain of f_λ. The proof relies on the work of Palais [13] on foliations.

This theorem generalizes that of Nomizu [10] who proved a similar result for hypersurfaces in the sphere with constant principal curvatures. Because of the abundance of examples of such hypersurfaces, one can produce (through stereographic projection) many examples of hypersurfaces in Euclidean and hyperbolic space to which our theorem applies (see §3.c).

If λ has constant multiplicity one, then $f_\lambda(M)$ is not an $(n-1)$-dimensional manifold without additional hypotheses. This case is handled by Theorem 3.2 which is a generalization of the classical determination of conditions under which a sheet of the focal set of a surface in E^3 is a curve. (See, for example, Eisenhart [6, p. 310-314].)

A key ingredient in the proofs of the above results is the computation of the rank of f_λ. Our result in this area (Theorem 2.1) applies to submanifolds of arbitrary codimension.

The classical version of these theorems was used by Banchoff [1] and Cecil [3] in characterizing taut immersions of surfaces in E^3. Applications of the results of the present paper to the classification of taut immersions of $S^k \times S^{n-k}$ into E^{n+1} may be found in our forthcoming paper [5].

1. Preliminaries. In this paper, all manifolds and maps are