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A CLASS OF ISOTROPIC DISTRIBUTIONS IN R_n AND THEIR CHARACTERISTIC FUNCTIONS

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Let $r(t)$ be a characteristic function. Suppose that there is an integer $n \geqq 2$ such that $r((t_1^2 + \cdots + t_n^2)^{1/2})$ is, as a function of n variables, also the characteristic function of some distribution in R_n. Then, as is known, the distribution is necessarily rotationally invariant, and r has a canonical form as a certain Bessel transform of a bounded nondecreasing function. A certain subclass of the class of such characteristic functions was defined and studied by Mittal, who furnished an analytic characterization of functions in the subclass. The purposes of this paper are (i) to present an alternative probabilistic characterization of these functions, and (ii) to characterize, for this subclass, the bounded nondecreasing function appearing in the Bessel transform.

1. Introduction and summary. Let $h(x)$, $x \geqq 0$, be a nonnegative function. For a fixed integer $n \geqq 2$, let $\boldsymbol{x}$ be an element of R_n, and let $\|\boldsymbol{x}\|$ be its Euclidian norm. Consider the extension of h to a radial function on R_n: $h(x) = h(\|\boldsymbol{x}\|)$, where $x = \|\boldsymbol{x}\|$. If $x^{n-1}h(x)$ is integrable over $x \geqq 0$, then $h(\|\boldsymbol{x}\|)$ is integrable over R_n, and

$$\int_{R_n} h(\|\boldsymbol{x}\|)d\boldsymbol{x} = S_n \int_0^\infty x^{n-1}h(x)dx ,$$

where S_n is the surface area of the n-dimensional unit sphere. If the integrals above have the value 1, then $h(\|\boldsymbol{x}\|)$ is a density function on R_n. Its characteristic function is, by definition,

$$\int_{R_n} \exp [i(\boldsymbol{t}, \boldsymbol{x})]h(\|\boldsymbol{x}\|)d\boldsymbol{x} , \tag{1.1}$$

and is a radial function, denoted $r(\|\boldsymbol{t}\|)$, $\boldsymbol{t} \in R_n$, where r is function of a real variable. According to Schoenberg's theorem [8], the function $r(t)$ is necessarily of the form

$$r(t) = \Gamma(n/2) \int_0^\infty \frac{J_{(n-2)/2}(ut)}{(ut/2)^{(n-2)/2}} dG(u) , \tag{1.2}$$

where J_ν is the Bessel function of order ν, and where G is a bounded, nondecreasing function of variation 1. The latter is directly related to h:

$$S_n h(x)x^{n-1}dx = dG(x) . \tag{1.3}$$