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REMARKS ON A THEOREM OF L. GREENBERG ON THE MODULAR GROUP

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Introduction. For integers a and b, each greater than 1, let $T(a, b)$ be the free product of cyclic groups of orders a and b. Then $T(a, b)$ has presentation

$$\langle X, Y: X^a = Y^b = 1\rangle .$$

Suppose that $G \lhd T(a, b)$. If XYG has finite order in $T(a, b)/G$, then the order is the *level* of G, denoted by $n(G)$. We put $U = XY$. When G has finite index $\mu(G)$, then $n(G)$ is defined, and divides $\mu(G)$. In such a case, $t(G) = \mu(G)/n(G)$ is the *parabolic class number* of G. These definitions agree with the usual ones for $T(2, 3)$, the classical modular group.

For $T(2, 3)$, Newman [7] raised the question of whether there were infinitely many normal subgroups with a given parabolic class number. In [3], L. Greenberg showed that this was not possible by proving that, for $t > 1$,

$$\mu \leqq t^4 .$$

Here, as later, we write μ, t for $\mu(G)$, $t(G)$ when the group is clear from the context.

Mason [5] improved this to

$$\mu \leqq t^3 . \tag{1}$$

This was also proved by Accola [1]. Implicit in his proof is a proof that (1) holds when a and b are distinct primes.

Here, we show that, when a and b are coprime, there is a constant $c(a, b)$ such that, for $t > 1$,

$$\mu \leqq c(a, b)t^2(t - 1) . \tag{2}$$

The constant is 1 when a and b are distinct primes, e.g., for the modular group. There is no corresponding result when a and b are not coprime.

We give examples to show that we can have equality in (2), but only a finite number of times for given a and b. Finally, we obtain a better result for large t.

The referee has drawn our attention to a paper of Morris Newman, '2-generator groups and parabolic class numbers', Proc. Amer. Math. Soc., 31 (1972), 51-53, which contains the weaker result