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AN ALGEBRA OF PSEUDO-DIFFERENTIAL OPERATORS WITH NON-SMOOTH SYMBOL

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In [4], [1], [7], and [5] certain algebras of zero-order pseudo-differential operators were discussed which all were generated by closing the operator algebra $\overset{\circ}{\mathfrak{A}}$ finitely generated from the elements

$$\{a(M),\ b(D)\colon\ a \in \mathscr{A}^{+},\ b \in \mathscr{A}^{\#}\}\ , \tag{0.1}$$

with multiplication operators $u(x) \to a(x)u(x)$ denoted by $a(M)$ and convolution operators (or formal Fourier multipliers) $b(D) = F^{*}a(M)F$, with $F =$ Fourier transform. Various classes $\mathscr{A}^{+}$ and $\mathscr{A}^{\#}$, and various operator topologies were used, with the purpose of using the generated topological algebra for proving normal solvability of singular elliptic problems $Lu = f$, $x \in \boldsymbol{R}^n$, with a suitable linear differential operator $L = \sum_{|\alpha| \leqq N} a_\alpha(x)D^\alpha$.

At present let us focus on the algebra $\mathfrak{A}_\infty$ obtained from the classes

$$\mathscr{A}^{+} = \{a \in C^\infty(\boldsymbol{R}^n)\colon a(x) = O(1),\ a^{(\beta)}(x) = o(1),\ \beta \neq 0\} \tag{0.2}$$

and

$$\mathscr{A}^{\#} = \{b \in C^\infty(\boldsymbol{R}^n)\colon\ b^{(\beta)} \in C(\boldsymbol{B}^n),\ \beta \in \boldsymbol{Z}^n_+\}\ , \tag{0.3}$$

with the compactification $\boldsymbol{B}^n$ of $\boldsymbol{R}^n$ obtained by continuous extension of the vector-valued function $x \to x(1 + x^2)^{-1/2}$, where we close under the following operator topology: $\overset{\circ}{\mathfrak{A}}$, with $\mathscr{A}^{+}$ and $\mathscr{A}^{\#}$ as in (0.2) and (0.3) may be seen to be a subalgebra of $\mathscr{L}(\mathfrak{H}_s)$, the algebra of continuous operators $\mathfrak{H}_s \to \mathfrak{H}_s$, with the L^2-Sobolev space $\mathfrak{H}_s = \{u\colon u \in \mathscr{S}',\ \|(1 - \Delta)^{s/2}u\|_{L^2} = \|u\|_s < \infty\}$ of $\boldsymbol{R}^n$. This is true for every $s \in \boldsymbol{R}$, and therefore the elements of $\overset{\circ}{\mathfrak{A}}$ also take the Frechet space $\mathfrak{H}_\infty$ continuously to itself. A locally convex topology on $\overset{\circ}{\mathfrak{A}}$ is generated by all the operator norms $\|A\|_s = \sup\{\|Au\|_s\colon \|u\|_s \leqq 1\}$. In fact this is a Frechet topology, and it suffices to only take the norms $\|A\|_k$, $k \in \boldsymbol{Z}$. All this is discussed in details in [2]. We define $\mathfrak{A}_\infty$ to be the completion of $\overset{\circ}{\mathfrak{A}}$ under that topology.

Similarly one may complete $\overset{\circ}{\mathfrak{A}}$ as a subalgebra of any given fixed $\mathscr{L}(\mathfrak{H}_s)$ in the norm topology, to obtain a Banach algebra $\mathfrak{A}_s$, which proves to be a C^*-subalgebra of $\mathscr{L}(\mathfrak{H}_s)$, containing the compact ideal $\mathfrak{K}_s = \mathfrak{K}(\mathfrak{H}_s)$ of $\mathscr{L}(\mathfrak{H}_s)$. In fact, $\mathfrak{A}_s/\mathfrak{K}_s$ is commutative, thus we have $\mathfrak{A}_s/\mathfrak{K}_s = C(\boldsymbol{M}_s)$, with a certain compact Hausdorff space $\boldsymbol{M}$, by the Gelfand-Naimark theorem. The space $\boldsymbol{M} = \boldsymbol{M}_s$ proves