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ALGEBRA HOMOMORPHISMS AND THE FUNCTIONAL CALCULUS

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Let b be a fixed element of a commutative Banach algebra with unit. Suppose $\sigma(b)$ has at most countably many connected components. We give necessary and sufficient conditions for b to possess a discontinuous functional calculus.

Throughout, let B be a commutative Banach algebra with unit 1 and let rad(B) denote the radical of B. Let b be a fixed element of B. Let $\mathscr{O}$ denote the LF space of germs of functions analytic in a neighborhood of $\sigma(b)$. By a functional calculus for b we mean an algebra homomorphism θ' from $\mathscr{O}$ to B such that $\theta'(z) = b$ and $\theta'(1) = 1$. We do not require θ' to be continuous. It is well.known that if θ' is continuous, then it is equal to θ, the usual functional calculus obtained by integration around contours i.e.,

$$\theta(f) = \frac{1}{2\pi i}\int_\Gamma f(t)(t-b)^{-1}dt\ ,$$

for $f \in \mathscr{O}$, Γ a contour about $\sigma(b)$ [1, I.4.8, Theorem 3]. In this paper we investigate the conditions under which a functional calculus θ' is necessarily continuous, i.e., when θ is the unique functional calculus.

In the first section we work with sufficient conditions. If S is any closed subspace of B such that $bS \subseteq S$, we let $D(b, S)$ denote the largest algebraic subspace of S satisfying $(b-\lambda)D(b, S) = D(b, S)$, all $\lambda \in \boldsymbol{C}$. We show that if θ' is a functional calculus for b and if we let $\beta \equiv \theta' - \theta$, then $\beta(\mathscr{O}) \subseteq D(b, \text{rad}(B))$. Hence if $D(b, B) \equiv (0)$, then $\theta = \theta'$. We show that this extends H. G. Dales earlier result that if rad(B) is finite dimensional, $\theta = \theta'$ [2, Theorem 1, application (a)].

In section two we seek converse results to the above. In general, if σ is a clopen subset of $\sigma(b)$, we let $E(\sigma)$ denote the projection $\theta(e(\sigma))$ where $e(\sigma)$ is one on σ and zero elsewhere. If τ is a connected component of $\sigma(b)$ we let

$$S(\tau) \equiv \bigcap_{\sigma\ \text{clopen},\ \sigma \supseteq \tau} E(\sigma)B\ ,$$

which is a closed ideal. We first show that if $D(b, S(\tau)) \not\equiv (0)$ for some connected component τ of $\sigma(b)$, then there exists a discontinuous functional calculus θ' for b. If we let $\beta = \theta' - \theta$ as before we may choose $\beta(\mathscr{O}) \subseteq D(b, S(\tau))$. We next show that if $\sigma(b)$ has only countably (or finitely) many components, then $D(b, B) \not\equiv (0)$ implies $D(b, S(\tau)) \not\equiv (0)$