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TYPE ANALYSIS OF THE REGULAR REPRESENTATION OF A NON-UNIMODULAR GROUP

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This paper is concerned with finding necessary and sufficient conditions for the von Neumann algebra $\mathscr{M}(G)$ generated by the left regular representation λ^G of a locally compact, separable, non-unimodular group G to be type I, semifinite, or to have a central summand of type III. In the case where the modular function δ_G of G has closed range, we are able to give a complete solution in terms of the orbit structure of the natural action of G on the reduced quasi-dual (Γ_H, μ_H) of the maximal unimodular subgroup $H = \text{kernel}\,\delta_G$. Thus $\mathscr{M}(G)$ is semifinite if and only if the action is smooth with isotropy subgroup H, and of type III_0 if and only if the action is completely nonsmooth. Conditions of a similar type are given which are necessary and sufficient for $\mathscr{M}(G)$ to have a summand of type III_λ, $\lambda \in (0, 1]$.

In §2 we develop the necessary preliminary material for the later work, establishing the connection between semidirect products of groups and crossed products of the corresponding group algebras. Sections 3 and 4 give the proof of the above mentioned criterion of semi-finiteness; this proof relies heavily on the theory of modular automorphisms, and crossed products as developed in [3], [20], and [21]. In §5 we turn to examples; we exhibit groups G_λ, $\lambda \in [0, 1]$ with $\mathscr{M}(G_\lambda)$ a factor of type III_λ, and also groups $G_{0,\lambda}$ with $\mathscr{M}(G_{0,\lambda})$ a factor of type III_0 with $T(\mathscr{M}(G_{0,\lambda})) = 2\pi/\log \lambda \boldsymbol{Z}$. In fact, we construct two such families of groups; one is a variation on Godements example of a group with type III regular representation, and for these groups the associated von Neumann algebras are not hyperfinite; the other family is constructed using the semi- direct product of an abelian group by a solvable group so that the associated von Neumann algebras are hyperfinite. This second family of examples is due to A. Connes (private communication). In the final section we use the results of §3 to give a form of the Plancherel theorem for locally compact separable groups G for which $\delta_G(G) = \boldsymbol{R}_+$ and $\mathscr{M}(G)$ is semifinite. For the most part this is an adaption of the more general formula in [18].

2. Preliminaries.

Throughout, G will denote a locally compact separable non-unimodular group, with modular function δ_G.

PROPOSITION 2.1. *Suppose $\delta_G(G) = \boldsymbol{R}_+$. Then there is a continuous one parameter subgroup $L = \{g_t : t \in \boldsymbol{R}\}$ of G with $\delta_G(g_t) = e^t$.*