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A NOTE ON EXPONENTIALS OF DISTRIBUTIONS

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Nonstandard analysis is used to discuss nonlinear functions of distributions. An application is given to obtain a generalized Trotter product formula. The strong resolvent topology is discussed from a nonstandard point of view.

Enlarging the real number system to include infinite and infinitesimal quantities enabled Laugwitz [5] to view the delta function distribution as a point function. Independently Robinson [7] demonstrated that distributions could be viewed as generalized polynomials. Luxemburg [6] presented an alternate picture of distributions as generalized functions within the context of Robinson's theory of nonstandard analysis and it is a special case of this point of view that we take here. Once one accepts distributions as generalized functions, the composition map provides a natural method of defining nonlinear functions of distributions. Unfortunately, the difficulties in the standard attempt to define a function, f, of a distribution, τ, (by first writing τ as a limit, in some appropriate sense, of smooth standard functions τ_n, next composing these approximations with f, and finally taking the limit of $f \circ \tau_n$) still remain in the nonstandard theory. In particular, this procedure may not lead to a distribution and even when it does the distribution obtained may depend on the representation of the original distribution as a generalized function. Thus, at present, there is no comprehensive theory of nonlinear functions of distributions.

The development of such a theory may well proceed along alternate tracks depending on the applications intended. One use for distributions occurs in the study of perturbations of selfadjoint operators in Hilbert space. In some perturbation problems the pathology of obtaining compositions which do not represent distributions may be avoided by considering only bounded functions of distributions. For example, in the Trotter product formula one works with bounded exponentials. Even though, in this case, the exponentials of distributions may be identified with standard distributions it is the conclusion of this paper that such an identification should not be made. It is the author's view that functions of distributions should be regarded as generalized functions but not distributions. The utility of this view is illustrated by a nonstandard version of the Trotter product formula.

For an introduction to nonstandard analysis and its relation to distributions, see [10]. In [11] a square root for the delta function