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COMBINATORIAL GEOMETRY AND ACTIONS OF COMPACT LIE GROUPS

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In this paper a theorem of combinatorial geometry will be applied to prove results about actions of compact Lie groups on manifolds.

In order to understand actions on differentiable manifolds, the weights of the tangential representations at fixed points of a maximal torus can be taken as basic data. Those weights are related by the structure of the equivariant cohomology ring of the manifold. The weights can also be considered as just a finite set of vectors or as a finite set of points in a projective space. From this point of view, theorems of combinatorial geometry can be used. Hence representation theory, equivariant cohomology theory, and combinatorial geometry can be used to understand differentiable actions. We will use the following result of combinatorial geometry which has been generalized by Sten Hansen [11]. It was conjectured by Sylvester[16] in 1893 and proved by Gallai in 1933.

THEOREM 1 (Sylvester-Gallai). *"Given a finite set of points in the real affine plane, there is a line containing exactly two of those points, unless the point set is collinear."*

It was during discussions with Ted Chang concerning his results in equivariant homotopy theory, that I realized that the theorem of Sylvester-Gallai was useful. I am grateful to Ted for explaining Theorem 5 to me. A very simple example of a result in equivariant homotopy theory follows.

THEOREM 2. *"Let a torus T be acting on S^{2n} and on S^{4n-1}, such that the action on S^{4n-1} is effective and such that $F(T, S^{4n-1}) = \varnothing$ and $F(T, S^{2n})$ is connected. If rank $T \geqq 4$, then every equivariant map $S^{4n-1} \to S^{2n}$ has trivial Hopf invariant." The attaching map $S^{15} \to S^{8}$ of the Cayley projective plane has nontrivial Hopf invariant and admits an effective action of a torus T of rank 3 with $F(T, S^{15}) = \varnothing$ and $F(T, S^{8}) = S^{2}$. Hence the bound rank $T \geqq 4$ cannot be relaxed.*

In general the results one can prove using combinatorial geometry say that a group acting on a specific manifold with a specific fixed point set cannot have rank exceeding a certain number. An