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ON THE THEORY OF COMPACT OPERATORS IN VON NEUMANN ALGEBRAS II

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In their recent works L. Zsido' and P. A. Fillmore have extended Weyl's version of the classical Weyl-von Neumann theorem to infinite semi-finite countably decomposable von Neumann factors, by proving that for every self-adjoint operator A in the factor there is a diagonal operator $B = \sum \lambda_n E_n$ such that $A - B$ is compact, the E_n are one-dimensional projections and $\{\lambda_n\}$ is dense in the essential spectrum of A. In this paper we extend the Weyl-von Neumann theorem in a different way.

First we extend the von Neumann version of the theorem to both finite and infinite factors by proving that $A - B$ can be chosen as a Hilbert-Schmidt operator of arbitrarily small norm. We have to drop the condition about the λ_n or the dimension of the E_n.

In the second section we shall first re-obtain an equivalent form of Fillmore's theorem and then we shall generalize it to the case of normal operators, thus extending the Berg-Sikonia-Halmos theorem (see [2], [13], and [8]) to infinite factors. Finally we shall examine the possibility of choosing B in the von Neumann algebra generated by A and we shall generalize to normal operators a connected theorem by Zsido' [15].

We wish to thank M. Sonis for having called our attention to this problem.

1. The Weyl-von Neumann theorem in von Neumann factors. Let H be a Hilbert space, $\mathscr{A}$ be a countably decomposable (i.e., σ finite) semi-finite (i.e., type I or II) von Neumann factor on H, $\mathscr{A}'$ be its commutant and $\mathscr{J}$ be the ideal of compact operators of $\mathscr{A}$, that is, the norm closure of the ideal of the operators $A \in \mathscr{A}$ with range projection R_A finite relatively to $\mathscr{A}$ (finite operators for short).

Let Tr (Tr') be a semi-finite faithful normal trace on $\mathscr{A}^+$ ($\mathscr{A}'^+$) and D (D') be its restriction to the projections of $\mathscr{A}$ ($\mathscr{A}'$). We use the normalization of the relative dimensions D and D' for which $D(I) = 1$ ($D'(I) = 1$) when $\mathscr{A}$ ($\mathscr{A}'$) is finite and the linking constant $C_{\mathscr{A}} = 1$ if $\mathscr{A}$, $\mathscr{A}'$ or both are infinite.

Let $\mathscr{S}_2(\mathscr{A})$ be the Hilbert-Schmidt class of $\mathscr{A}$, i.e., the (generally incomplete) normed ideal of the operators $A \in \mathscr{A}$ for which