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CARDINAL INEQUALITIES FOR TOPOLOGICAL SPACES INVOLVING THE WEAK LINDELOF NUMBER

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Let $wL(X)$, $\chi(X)$, $\psi(X)$, $c(X)$, and $\partial(X)$ denote respectively the weak Lindelof number, character pseudocharacter, cellularity, and tightness of a Hausdorff topological space X. It is proved that if X is a normal Hausdorff space then $|X| \leqq 2^{\chi(X)wL(X)}$. Examples are given of a nonregular Hausdorff space Z such that $|Z| > 2^{\chi(Z)wL(Z)}$ and a zero-dimensional Hausdorff space Y such that $|Y| > 2^{\psi(Y)\partial(Y)wL(Y)}$. Define $r\psi(X) =$ min $\{\kappa$: each closed subset of X is the intersection of the closures of κ of its neighborhoods$\}$. It is proved that $c(X) \leqq r\psi(X)wL(X)$. Related open questions are posed.

1. Introduction. Let X be a Hausdorff topological space. The weak Lindelof number of X, denoted $wL(X)$, is defined to be min $\{\kappa$: each open cover of X has a subfamily of cardinality no greater than κ whose union is a dense subspace of $X\}$. If $wL(X) = \aleph_0$ we say that X is weakly Lindelof; see [9] and [10] for information concerning these spaces. It is immediate that $wL(X) \leqq L(X)$, where $L(X)$ is the Lindelof number of X (definitions of this and other cardinal functions are given below). It is only slightly less trivial to see that $wL(X) \leqq c(X)$, where $c(X)$ denotes the cellularity of X. The theme of this paper is the study of situation in which $L(X)$ and/or $c(X)$ can be replaced, in inequalities involving cardinal functions on X, by $wL(X)$. We also consider in detail several illuminating counterexamples which place bounds on the situations in which such substitutions can be made.

Perhaps the most famous inequality involving cardinal functions is Arhangel'skii's theorem [2], which answered a fifty year old question of Alexandroff. It asserts that if X is a Hausdorff space then $|X| \leqq 2^{\chi(X)L(X)}$, where $\chi(X)$ denotes the character of X. It has also been proved that if X is Hausdorff then $|X| \leqq 2^{\chi(X)c(X)}$ (see [4]). One is led to conjecture that the common generalization of these theorems is true, namely that $|X| \leqq 2^{\chi(X)wL(X)}$. In Theorem 2.1 we prove that if X is a normal Hausdorff space this is true; the proof is a modification of a technique used by Pol [6] to give an elegant proof of Arhangel'skii's theorem. We then demonstrate the need for some separation axioms in the hypotheses of this theorem by exhibiting two examples; first a nonregular Hausdorff space Z for which $|Z| > 2^{\chi(Z)wL(Z)}$, and second a zero-dimensional Hausdorff space Y for which $|Y| > 2^{\psi(Y)\partial(Y)wL(Y)}$ (here $\psi(Y)$ and $\partial(Y)$ denote respec-