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A NOTE ON A CONJECTURE OF HECKE

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In 1940 Hecke made a conjecture concerning the representation of modular forms of weight 2 on $\Gamma_0(p)$, p a prime, by theta series. In this note we show that this conjecture is true if and only if $p < 37$ or $p = 41, 47, 59,$ or 71. We also propose a modified version of the conjecture which we have tested quite extensively.

1. Introduction. Hecke's conjecture (see [7], Satz 53 p. 884) states that a certain explicit set of theta series coming from quaternion algebras can be used to give a basis for the space of cusp forms of weight 2 on $\Gamma_0(p)$. Eichler knew in 1956 that Hecke's conjecture did not hold in general (see [3], p. 169). However, since there has been quite a bit of confusion on this point (see [5], p. 138 and [6], p. 148–9), we think it is worthwhile to reconsider the conjecture. More importantly, in Question 3.6 below, we propose a modified version of Hecke's conjecture which we have tested quite extensively on a computer.

Fix a prime p and let $\mathfrak{A}$ denote the (unique definite) quaternion division algebra over Q, the field of rational numbers, ramified precisely at p and ∞. Let $\mathcal{O}$ be a maximal order of $\mathfrak{A}$ and let I be a left $\mathcal{O}$-ideal (see [5], Ch. 2 or [11]). We define the theta series $\theta_I(\tau)$, determined by I, by

$$\Theta_I(\tau) = \sum_{z \in I} e^{2\pi i \tau N(z)/N(I)} \tag{1}$$

where $N(\ )$ denotes the reduced norm of $\mathfrak{A}$ and $N(I)$ is the positive rational number that generates the fractional ideal of Q generated by $\{N(a) \mid a \in I\}$. It is well known (see e.g., [5]) that $\Theta_I(\tau)$ is a modular form of weight 2 on $\Gamma_0(p)$. Let us denote by $S_k(\Gamma_0(p))$ the space of cusp forms of weight k on $\Gamma_0(p)$.

Let $\mathcal{O}$ be a maximal order of $\mathfrak{A}$ and let $I_1, \cdots, I_H$ be a complete set of representatives of the distinct left $\mathcal{O}$-ideal classes (we say two left $\mathcal{O}$-ideals I and J belong to the same class if and only if $I = Ja$ for some $a \in \mathfrak{A}^x$). It is well known that $H = H_p$, the *class number*, is finite and independent of the particular maximal order we choose (see [5] and [11]). Given two theta series $\Theta_I(\tau)$ and $\Theta_J(\tau)$, I and J left $\mathcal{O}$-ideals, then $\Theta_I(\tau) - \Theta_J(\tau)$ is a cusp form (see Siegel [15], p. 376). Hecke's conjecture states that for any maximal order $\mathcal{O}$ and representatives $I_1, \cdots, I_H$ of the left $\mathcal{O}$-ideal classes, $\Theta_{I_2}(\tau) - \Theta_{I_1}(\tau), \cdots, \Theta_{I_H}(\tau) - \Theta_{I_1}(\tau)$ constitute a basis for $S_2(\Gamma_0(p))$. One