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IMAGES OF SK_1ZG

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The computation of SK_1ZG for finite nonabelian groups G remains a difficult problem. Few examples are known in which SK_1ZG is nontrivial. One way to uncover nontrivial elements is to examine the homomorphic images of SK_1ZG under $K_1(-)$ of ring maps $ZG \to \Lambda$. Such images are investigated here in the cases where Λ is a commutative ring, a noncommutative order or a semisimple artinian image of ZG. Even trivial images illuminate the structure of SK_1ZG through K-theory exact sequences.

2. Terminology. The word "ring" refers to an associative ring with an identity. The group of units of a ring Λ is denoted Λ^*. "Map" meas homomorphism. Unless otherwise specified, G denotes a finite group, and R, the ring of integers in an algebraic number field F.

3. The origin of K_1. In 1950 J. H. C. Whitehead introduced the notion of simple homotopy equivalence. A natural question in this theory is the following: Which CW-complexes are equivalent, relative to a common subcomplex L, under deformations which add and delete cells in a "simple" way along the cell structure? (See [3] for details.)

The answer lies in the computation of the Whitehead group $Wh(L)$, which depends only on the fundamental group $\pi_1(L)$. In fact it is obtained by the following algebraic construction: Let ZG denote the integral group ring of a (possibly infinite) group G. Then $GL(ZG)$ is the group of all invertible matrices over ZG, with matrices A and B identified if $A = \begin{bmatrix} B & 0 \\ 0 & I_n \end{bmatrix}$ for some size identity matrix I_n. The commutator subgroup $E(ZG)$ of $GL(ZG)$ is generated by all elementary matrices, obtained from the identity by adding a ZG multiple of one row to another. The quotient $GL(ZG)/E(ZG)$ is written K_1ZG. The trivial units $\pm G$ of ZG are 1×1 matrices in $GL(ZG)$. The Whitehead group $Wh_1(G)$ of G is $K_1ZG/\text{Image}\,(\pm G)$. If L is a CW-complex, $Wh(L) = Wh_1(\pi_1(L))$.

Group maps $G \to H$ extend to ring maps $ZG \to ZH$, and entry-wise on representative matrices to groups $K_1ZG \to K_1ZH$. This makes $K_1Z(-)$ a functor from groups to abelian groups. Replacing ZG, the same construction provides the functor $K_1(-)$ from rings to abelian groups. However, the group G plays a special role in the computation of K_1ZG, which apparently has no natural analog in $K_1\Lambda$ for an arbitrary ring Λ.