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MULTIFUNCTIONS AND GRAPHS

JAMES E. JOSEPH

In this paper we introduce the notions of multifunctions with strongly-closed graphs and θ-closed graphs. We then extend these notions as well as the notion of multifunctions with closed graphs and use these extensions to generalize and extend various known results on multifunctions. These generalizations and extensions include a number of sufficient conditions for multifunctions to be upper-semi-continuous, several generalizations and extensions of the well-known Uniform Boundedness Principle from analysis, and several "fixed set" theorems.

A *multifunction* $\Phi: X \to Y$ is a correspondence from X to Y with $\Phi(x)$ a nonempty subset of Y for each $x \in X$. We will denote the graph of Φ, i.e., $\{(x, y): x \in X \text{ and } y \in \Phi(x)\}$, by $G(\Phi)$. As usual, if X and Y are topological spaces (hereafter referred to as "spaces") and $\Phi: X \to Y$ is a multifunction we will say that Φ has a *closed graph* if $G(\Phi)$ is a closed subset of the product space $X \times Y$. If X and Y are spaces a multifunction $\Phi: X \to Y$ is said to be *upper-semi-continuous* (*u.s.c.*) *at* $x \in X$ if for each W open about $\Phi(x)$ in Y there is a V open about x in X with $\Phi(V) \subset W$; Φ is said to be *upper-semi-continuous* (*u.s.c.*) if Φ is u.s.c. at each $x \in X$. It is not difficult to establish that Φ is u.s.c. if and only if $\Phi^{-1}(K)$ is closed in X whenever K is closed in Y. Smithson [20] has given a survey of some of the principal results on multifunctions. See [20] and [8] for definitions not given here.

2. Some preliminary definitions and theorems. We will denote the closure of a subset K of a space by $\mathrm{cl}(K)$, the adherence of a filterbase Ω on the space by $\mathrm{ad}\,\Omega$, and the family of open subsets which contain K by $\Sigma(K)$. A point x is in the *θ-closure of a subset* K of a space ($x \in \mathrm{cl}_\theta(K)$) if each $V \in \Sigma(x)$ satisfies $K \cap \mathrm{cl}(V) \neq \varnothing$; K is *θ-closed* if $\mathrm{cl}_\theta(K) \subset K$; x is in the *θ-adherence of a filterbase* Ω on the space ($x \in \mathrm{ad}_\theta\,\Omega$) if $x \in \mathrm{cl}_\theta(F)$ for each $F \in \Omega$; a filterbase Ω on a space *θ-converges* to a point x in the space ($\Omega \to_\theta x$) if for each $V \in \Sigma(x)$ there is an $F \in \Omega$ with $F \subset \mathrm{cl}(V)$ [25]. In [23], a space is called an *$H(i)$ space* if each open filterbase on the space has a nonempty adherence. Hausdorff $H(i)$ spaces are called *H-closed*. A subset A of a space X is defined to be $H(i)$ if and only if each filterbase Ω on A satisfies $A \cap \mathrm{ad}_\theta\,\Omega \neq \varnothing$ [8].

We will say that a multifunction $\Phi: X \to Y$ has *closed* (*θ-closed*) [*compact*] *point images* if $\Phi(x)$ is closed (θ-closed) [compact] in Y for