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MODULARITY OF THE CONGRUENCE LATTICE OF A COMMUTATIVE CANCELLATIVE SEMIGROUP

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Modularity of the lattice of normal subgroups of a group is well-known. Equivalently, the lattice of congruence relations on a group is a modular lattice. A natural question to consider is how far can we push the last statement when dealing with the larger class semigroups. It is easily shown that the class of congruence lattices of semigroups satisfies no nontrivial lattice identity. Thus we might try to find those semigroups whose congruence lattice is a modular lattice. This problem is of all the more interest due to the fact that congruences on algebras whose congruence lattice is a modular lattice satisfy variants of the Jordan-Holder-Schreier theorem. In this paper we show that the commutative cancellative semigroups whose congruence lattice is a modular lattice are the abelian groups, the positive cones of rational groups, and the nonnegative cones of rational groups. We also show that the commutative cancellative semigroups with a distributive lattice of congruences are locally cyclic or locally cyclic with an identity adjoined. This last result generalizes Ore's theorem that a group has a distributive lattice of congruences if and only if it is locally cyclic.

1. Introduction. By an N-semigroup we mean a commutative cancellative archimedean semigroup without idempotent. An $\bar{N}$-semigroup will denote a commutative cancellative idempotent free semigroup (CCIF-semigroup) which contains an ideal which is an N-semigroup.

It was shown in 1957 by Tamura [14] (see also [1] and [13]) that every N-semigroup S can be represented by an abelian group G and a function I from $G \times G$ into the nonnegative integers N^0 with the properties

(1.1) $$I(g, h) = I(h, g) \quad \text{for all} \quad g, h \in G ,$$

(1.2) $$I(g, h) + I(gh, k) = I(g, hk) + I(h, k) \quad \text{for all} \quad g, h, k \in G ,$$

(1.3) $I(g, e) = 1$ for all $g \in G$, where e is the identity of G ,

(1.4) For each $g \in G$ there exists $m > 0$ such that $I(g, g^m) > 0$.

Where $S = N^0 \times G$ with the product

(1.5) $$(m, g)(n, h) = (m + n + I(g, h), gh) \text{ for } (m, g), (n, h) \in N^0 \times G .$$