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A CONVOLUTION RELATED TO GOLOMB'S ROOT FUNCTION

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The root function $\gamma(n)$ is defined by Golomb for $n>1$ as the number of distinct representations $n=a^b$ with positive integers a and b. In this paper we define a convolution ∇ such that γ is the ∇-analog of the (Dirichlet) divisor function τ. The structure of the ring of arithmetic functions under addition and ∇ is discussed. We compute and interpret ∇-analogs of the Moebius function and Euler's Φ-function. Formulas and an algorithm for computing the number of distinct representations of an integer $n\geqq 2$ in the form $n=a_1^{a_2^{\cdot^{\cdot^{\cdot^{a_k}}}}}$, with a_i a positive integer, $i=1,\cdots,k$, are given.

1. Introduction. Let Z denote the set of positive integers, let A denote the set of arithmetic functions (complex-valued functions with domain Z), and let F denote the set of elements of Z which are not kth powers of any positive integer for $k>1 (k\in Z)$. Note that $1\notin F$. The divisor function τ can be defined as $\tau=\nu_0*\nu_0$, where $\nu_0\in A$, $\nu_0(n)=1$ for all $n\in Z$, and $*$ is the Dirichlet convolution defined for $\alpha,\beta\in A$ by $(\alpha*\beta)(n)=\sum_{d|n}\alpha(d)\beta(n/d)$.

Any integer $n\geqq 2$ having canonical form $n=p_1^{e_1}\cdots p_r^{e_r}$ is uniquely expressible as $n=m^g$, where $g=g.c.d.\ (e_1,\cdots,e_r)$ and $m\in F$. Golomb [1] defines the root function $\gamma(n)$ for $n\in Z, n>1$, as the number of distinct representations $n=a^b$ with $a,b\in Z$; and he notes that $\gamma(n)=\tau(g)$ for $n=m^g$, $m\in F$, $g\in Z$. We let $\gamma(1)=1$.

For $\alpha,\beta\in A$, $n=m^g$, with $m\in F, g\in Z$, we define the G-convolution ("Golomb" convolution), ∇, by

$$(1.1)\qquad (\alpha\nabla\beta)(n)=\sum_{d|g}\alpha(m^d)\beta(m^{g/d})\ .$$

We define $(\alpha\nabla\beta)(1)=1$. This G-convolution is not of the Narkiewicz type [2, 4].

In § 2, we show that $\{A,+,\nabla\}$ (where $(\alpha+\beta)(n)=\alpha(n)+\beta(n)$, $n\in Z$) is a commutative ring with unity and we characterize the units and the divisors of zero. We define a G-multiplicative function and note that the set of G-multiplicative units in $\{A,+,\nabla\}$ forms an Abelian group under the operation ∇.

We choose to define ∇ as in (1.1) because then $(\nu_0\nabla\nu_0)(n)$ equals $\gamma(n)$, the number of distinct representations of n as a^b, $a,b\in Z$;