PACIFIC JOURNAL OF MATHEMATICS
Vol. 79, No. 2, 1978

QUASI-ADDITIVITY AND SETS OF FINITE L^p-CAPACITY

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The Bessel L^p-capacity of order $\alpha>0$, $B_{\alpha,p}$, and the Riesz L^p-capacity of order α, $R_{\alpha,p}$, are shown to have the same sets of finite capacity in Euclidean R^n, $\alpha p<n$. However, they have markedly different behavior as countably "almost" additive (quasi-additive) set functions - i.e., as applied to sets that are partitioned by increasing concentric rings.

There are several useful versions of an L^p-capacity (defined on subsets of R^n) in the literature. They are all more or less direct generalizations of the classical notion of capacity based on Laplace's equation in two and three dimensions (i.e., the capacity used by N. Wiener et al.) which corresponds to the case $p=2$, $\alpha=1$ below. We will be interested here in two important canonical examples that have attracted some attention of late: the L^p-Bessel capacity, $B_{\alpha,p}$, and the L^p-Riesz capacity, $R_{\alpha,p}$. These set functions are quite useful in the function theory for the Sobolev spaces and in the theory of partial differential equations. Most of these applications require either a detailed knowledge of the nature of the exceptional sets (the sets of capacity zero) or estimates on the rate at which the capacity of a sequence of bounded sets tends to zero (e.g., in the Wiener criteria). In either case, it is local information that is being sought, either about the capacities themselves or about the potentials used to define them.

But these capacities give global information as well, especially with regard to the existence of certain Sobolev functions and solutions to certain elliptic partial differential equations. For example, a subset $A\subset R^n$ has finite Bessel capacity $B_{\alpha,p}$ iff there is a Sobolev function $u\in W^{\alpha,p}(R^n)$ (i.e., $u\in L^p(R^n)$ and $D^\alpha u\in L^p(R^n)$, D^α denoting all derivatives of order α) such that $u\equiv 1$ q.e. (quasi-everywhere) on A. Or if Ω is an open set of R^n with $R_{1,p}(\tilde{\Omega})<\infty$, $\tilde{\Omega}=$ complement of Ω, $1<p<n$, then there is a solution to the Euler equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u)=0$ in Ω which is equal to one q.e. on $\tilde{\Omega}$ and with $\int_\Omega|\nabla u|^p dx<\infty$ — the "equilibrium potential." These examples are extended to more general boundary values in [1].

In this note, however, we are interested in two special global properties of the capacities R and B: (a) the sets in R^n of finite capacity and (b) quasi-additivity.

(a_1) Our first result shows that both $R_{\alpha,p}$ and $B_{\alpha,p}$ have the same sets of finite capacity, $\alpha p<n$. Indeed, the inequality $R_{\alpha,p}(A)\leqq Q\cdot B_{\alpha,p}(A)$, with Q independent of A, is elementary, but the reverse