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APPROXIMATION BY RATIONAL MODULES ON NOWHERE DENSE SETS

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Let X be a compact subset of the complex plane. Let the module $\mathscr{R}(X)\bar{\mathscr{P}}_m$ be the space of all functions of the form

$$r_0(z) + r_1(z)\bar{z} + \cdots + r_m\bar{z}^m$$

where each r_i is a rational function with poles off X. We prove that $\mathscr{R}(X)\bar{\mathscr{P}}_1$ is dese in $L^p(X)$ for all $1 \leqq p < \infty$ and $\mathscr{R}(X)\bar{\mathscr{P}}_2$ is dense in $\mathscr{C}(X)$ if X has no interior point. As corollaries, we also prove that $\mathscr{R}(X)\bar{\mathscr{P}}_2$ is dense in lip (α, X) for all $0<\alpha<1$ and $\mathscr{R}(X)\bar{\mathscr{P}}_3$ is dense in $D^1(X)$ for the same X.

1. Introductions. Let X be a compact subset of the complex plane. Let the module $\mathscr{R}(X)\bar{\mathscr{P}}_m$ be the space $\mathscr{R} + \mathscr{R}\bar{z} + \cdots + \mathscr{R}\bar{z}^m$

$$= \{r_0(z) + r_1(z)\bar{z} + \cdots + r_m(z)\bar{z}^m\} ,$$

where each r_i is a rational function with poles off X. In [3, 4], O'Farrell has studied the relation of the problems of approximation by rational modules in different Lipschitz norms, and in the uniform norm, etc., to one another.

In this note, we investigate the problem of determining the set X so that $\mathscr{R}(X)\bar{\mathscr{P}}_m$ is uniformly dense in $\mathscr{C}(X)$ for each m.

Vitushkin [8] has given a necessary and sufficient condition in terms of analytic capacities for the case $m = 0$. In [3], O'Farrell has given an example of an X such that $\mathscr{R}(X)\bar{\mathscr{P}}_1$ is uniformly dense in $\mathscr{C}(X)$ whereas $\mathscr{R}(X)$ fails to be dense in $\mathscr{C}(X)$.

It is apparent that if X has interior, then $\mathscr{R}(X)\bar{\mathscr{P}}_m$ can not be dense in $\mathscr{C}(X)$. Thus we restrict our attention to a compact set X without interior throughout this note. Let $L^p(X) = L^p(\chi_X dm)$, where dm denotes the 2-dimensional Lebesgue measure. We prove the following theorem:

THEOREM. *Let X be a compact set with no interior. Then*

I. *$\mathscr{R}(X)\bar{\mathscr{P}}_1$ is dense in $L^p(X)$ for all $1 \leqq p < \infty$, and*

II. *$\mathscr{R}(X)\bar{\mathscr{P}}_2$ is dense in $\mathscr{C}(X)$.*

2. *Proof of theorem.* Let μ be a (finite Borel) measure on X. The Cauchy transform $\hat{\mu}$ is defined by