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$H^2(\mu)$ SPACES AND BOUNDED POINT EVALUATIONS

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Let $H^2(\mu)$ denote the closure of the polynomials in $L^2(\mu)$, where μ is a positive finite compactly supported Borel measure carried by the closed unit disc $\bar{D}$. For $\lambda \in \bar{D}$, define $E(\lambda)=\sup\{|p(\lambda)|/||p||_\mu\}$, where the suprenum is taken over all polynomials whose $L^2(\mu)$ norm is not zero. If $E(\lambda)<\infty$ we say that μ has a bounded point evaluation at λ, abbreviated b.p.e. at λ. Whenever $E(\lambda)<\infty$ we may fix the value of $f \in H^2(\mu)$ at λ. We determine the set on which all functions in $H^2(\mu)$ have (fixed) analytic values in terms of the parts of the spectrum of a certain operator.

In the case that the support of μ has a hole H bounded by an exposed arc Γ contained in ∂D and $E(z)$ is finite in H, we show how to recover the absolutely continuous part (with respect to Lebesgue measure on ∂D) of $d\mu|_\Gamma$ from a knowledge of the $E(z)$'s in H. A corollary of this is that for such measures μ the functions in $H^2(\mu)$ behave locally near Γ like those of classical Hardy space. That is, they have boundary values and their zero sets near Γ satisfy a Blaschke type growth condition. We apply this corollary to measures of the form $d\nu = GdA + wd\sigma$ to study the local behavior of functions in $H^2(\nu)$ near Γ (A denotes planar measure on $\bar{D}$, $d\sigma$ denotes linear Lebesgue measure on ∂D, and G and w are in an appropriate sense not too small on D and Γ respectively).

1. Bounded evaluations and analytic extensions of functions in $H^2(\mu)$. Let μ be a finite positive compactly supported Borel measure carried by the closed unit disc $\bar{D}$. We note that for λ a complex number, the point evaluation functional defined on polynomials by

$$p \longrightarrow p(\lambda)$$

is bounded with respect to the $L^2(\mu)$ norm if and only if $E(\lambda)<\infty$. In this latter case, by the Riesz representation theorem there is a unique element of $H^2(\mu)$, denoted by k_λ, satisfying

$$p(\lambda) = \langle p, k_\lambda \rangle$$

for all polynomials p and $||k_\lambda|| = E(\lambda)$. We call k_λ the bounded evaluation functional for μ at λ, abbreviated b.e.f. for μ at λ.

If μ has a b.p.e. at λ with b.e.f. k_λ and $f \in H^2(\mu)$, then we fix the value of f at λ by