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THE GROUP-VALUED LEBESGUE DECOMPOSITION

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An s-bounded additive map on a topological Boolean algebra to a topological group can be decomposed into a continuous and a singular part. This can be done in a canonical way as a limit thoerem in spaces of oprators. As a consequence, if $\mathscr{A}$ is a Boolean algebra of continuous projections on a (complete) topological group X and $\mathscr{G}$ is a "Fréchet-Nikodým" topology on $\mathscr{A}$, then every x in X, viewed as an additive map $A \to Ax$ on $\mathscr{A}$, can be decomposed uniquely as the sum of a $\mathscr{G}$-continuous and a $\mathscr{G}$-singular part. If $\mathscr{A}$ is equicontinuous, the operators which decompose x are continuous. The result applies to the space of all s-bounded additive functions on an algebra of sets to a complete separated topological group.

In 1963, R. B. Darst [4] published a proof of a Lebesgue-type decomposition theorem for complete normed abelian groups. The result extended C. E. Rickart's theorem [12] on decomposition of Banach-space valued set functions. It also turned out to contain subsequent results of J. J. Uhl [14] and of J. K. Brooks [2]. This was shown in detail by T. P. Dence [6]. About the same time, Darst [5] indicated how his work carried over to lattices of projection operators, and hence to modular functions defined on a lattice of sets. The purpose of this note is to show how the results and methods of [13] extend and shed light on this and related work. In particular, we find that the topology on the group need not be metrizable and that the same goes for the notion of convergence on the algebra (it is metrizable in Darst's work). (In passing, we see that a hypothesis of Darst's result can be removed so that it applies to join semilattices.) The results may be formulated as limit theorems in spaces of operators, so as to apply to all "decomposable" elements simultaneously.

Let X be a commutative separated topological group and $\mathscr{A}$ a Boolean algebra (with operations noted by the corresponding set theoretic symbols). Put a topology $\mathscr{G}$ on $\mathscr{A}$ such that $AB = A \cap B$ is continuous in A uniformly for B in $\mathscr{A}$ and that $A \triangle B$ is jointly continuous in A and B (an *FN-topology* in the sense of Drewnowski [8]). A map m on $\mathscr{A}$ to X is called $\mathscr{G}$-continuous if it is continuous in the topology $\mathscr{G}$ and is called $\mathscr{G}$-singular if for each neighborhood U of 0 in X and each $\mathscr{G}$-neighborhood G of 0 in $\mathscr{A}$, there exists A in G with $m((A^c)) = \{m(A^cE): E \in \mathscr{A}\} \subset U$. m is called sideways continuous [1](=s-bounded, =exhaustive), provided $mA_i \to 0$, whenever