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A LOWER BOUND FOR THE NUMBER OF CONJUGACY CLASSES IN A FINITE NILPOTENT GROUP

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A lower bound is given for the number of conjugacy classes in a finite nilpotent group which reflects the nilpotency class of the group.

The problem of estimating the number of conjugacy classes, k, in a finite group G, has been around since the turn of the century. Probably the earliest version of the problem is the question: Do there exist groups of arbitrarily large finite order with a fixed number of conjugacy classes? In 1903 Landau [4] answered this question in the negative by showing $k(G)$ goes to infinity with $|G|$. By refining Landau's technique, Erdos and Turan [2] proved $k(G) > \log_2 \log_2 |G|$. The known lower bound for $k(G)$ when G is nilpotent is somewhat better, $k(G) > \log_2 |G|$. This follows from a parametric equation for $k(G)$ when G is a p-group given by Poland [5].

In [3] Gustafson posed the problem of finding improved lower bounds for $k(G)$. Recently, Bertram [1] provided a substantial improvement of the $\log_2 \log_2 |G|$ bound which holds for "most" group orders. The purpose of this note is to give a lower bound for $k(G)$ when G is nilpotent which reflects the nilpotency class of G and improves the $\log_2 |G|$ bound.

THEOREM. *If G is a finite nilpotent group of nilpotency class n, then $k(G) \geqq n|G|^{1/n} - n + 1$.*

Proof. We observe that

$$G = Z_0 \cup \left(\bigcup_{i=1}^{n} Z_i - Z_{i-1}\right) \tag{1}$$

where $e = Z_0 \subsetneqq Z_1 \subsetneqq \cdots \subsetneqq Z_n = G$ is the upper central series of G. Since Z_i and Z_{i-1} are normal subsets of G, $Z_i - Z_{i-1}$ is a union of conjugacy classes of G. Indeed, for $x \in Z_i - Z_{i-1}$ and $g \in G$ we have $x^{-1}g^{-1}xg \in Z_{i-1}$ because Z_i/Z_{i-1} is the center of G/Z_{i-1}. This implies $g^{-1}xg \in xZ_{i-1}$ and we conclude $\bar{x}$, the conjugacy class of x in G, is contained in xZ_{i-1}. Thus $|\bar{x}| \leqq |xZ_{i-1}| = |Z_{i-1}|$ and therefore $Z_i - Z_{i-1}$ is a union of at least $|Z_i|/|Z_{i-1}| - 1$ conjugacy classes. It follows from (1) that

$$k(G) \geqq 1 + \sum_{i=1}^{n} (|Z_i|/|Z_{i-1}| - 1)$$