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TRANSVERSE WHITEHEAD TRIANGULATIONS

MARTIN SCHARLEMANN

Suppose M and N are PL manifolds and $f: M \to N$ is a proper PL map. Triangulate M and N so that f is simplical and let X be the dual complex in N. Then for each open simplex σ in X, $f^{-1}(\sigma)$ is a PL submanifold of M, so the stratification of N by the open simplices of X pulls back to a stratification of M. In other words, any such PL map can be regarded as a map of combinatorially stratified sets in which each n-stratum of therange is a disjoint union of copies of R^n. Here we prove the analogous theorem for a smooth map $f: M \to N$ between smooth manifolds.

An essentially similar (but simplified, since 1.1 is obvious) version of our proof would also apply to PL maps between PL manifolds, so our main theorem applies in the PL category as well. The theorem will be used elsewhere [2] to show that Cohen's notion of transverse cellularity [1] may be applied in the smooth category as well.

Let N be a smooth n-manifold imbedded in some high-dimensional Euclidean space R^N. An imbedding $h: X \to N$ of a simplical complex X into N is called a smooth imbedding if $h^{-1}(\partial N)$ is a subcomplex and, for every k-simplex σ of X, there is a neighborhood U of $h(\sigma)$ in R^N and a diffeomorphism $g: U \to R^k \times R^{N-k}$ such that $gh: \sigma \to R^k \times \{0\}$ is the linear map of σ onto the standard k-simplex $\Delta^k \subset R^k$. (The use of R^N is solely to avoid a special discussion of ∂N.) If X is a combinatorial manifold and h is a homeomorphism, then h is called a smooth triangulation of N. Combinatorial triangulations of smooth manifolds always exist (see e.g., [4]).

If $f: M \to N$ is a smooth map of manifolds, a smoothly imbedded complex $h: X \to N$ is said to be transverse to f over a closed k-simplex σ in X if the composition $p_2 gf: M \to R^{N-k}$ has no critical points near $f^{-1}(h(\sigma))$. In particular, $f^{-1}(h(\mathring{\sigma}))$ is a smooth submanifold of M. The definition is independent of the choice of U, g, or the imbedding of N in R^N. Our goal is the proof of

THEOREM 0.1. *Let $f: M \to N$ be a proper smooth map of smooth manifolds, $X \subset N$ a smoothly imbedded simplicial complex, $K \subset X$ a subcomplex of X transverse to f.*

If $X \cap \partial N \subset K$ or ∂N is transverse to f then there is an ambient diffeotopy $h_t: N \to N$, fixed near K, from the identity h_0 to a map h_1 such that $h_1(X)$ is transverse to f. Moreover, the diffeotopy h_t