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SOME PROPERTIES OF THE CHEBYSHEV METHOD

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Several properties of the Chebyshev method of summability, defined by G. G. Bilodeau, are investigated. Specifically, it is shown that the Chebyshev method is translative and is a Gronwall method. It is shown that the de Vallee Poussin method is stronger than the Chebyshev method, and that the Chebyshev method is not stronger than the $(C, 1)$ method. The final result shows that the Chebyshev method exhibits the Gibbs phenomenon.

Let $\Sigma(-1)^i u_i$ be an alternating series with partial sums $s_n = \sum_{i=0}^{n}(-1)^i u_i$. Define a sequence of polynomials $\{P_n(t)\}$ by $P_n(t) = \sum_{k=0}^{n} a_{nk}t^k$, $P_n(1) = 1$, $n = 0, 1, 2, \cdots$. The series $\Sigma(-1)^i u_i$ will be called summable (P_n) to the value s if $\lim \sigma(P_n) = s$, where $\sigma(P_n) = \sum_{k=0}^{n} a_{nk}s_k$. Bilodeau [1] considered the following question. What are sufficient conditions on P_n for $\sigma(P_n)$ to speed up the rate of convergence of a convergent sequence $\{s_n\}$? For sequences $\{u_n\}$ which are moment sequences, i.e., u_n has the representation $u_n = \int_0^1 t_n d\alpha(t)$, where $\alpha(t) \in BV[0, 1]$, he obtains the estimate $|\sigma(P_n) - s|/|r_n| \leqq (\mu_n/|r_n|)\int_0^1 t(1+t)^{-1}|d\alpha(t)|$, where $s = \sum_{i=0}^{\infty}(-1)^i u_i$, $r_n = s_n - s$, and $\mu_n = \max_{0\leqq t\leqq 1}|P_n(-t)|$. Adopting μ_n as a measure of the value of the method $\sigma(P_n)$, the most desirable sequence of polynomials will be those for which μ_n is a minimum, subject to the constraint $P_n(1) = 1$ for each n. The Chebyshev polynomials, defined by $T_n(x) = \cos nx$, $n = 0, 1, 2, \cdots, x = \cos\theta$, form the best approximation to the zero function over the interval $[-1, 1]$. When translated to $[0, 1]$ they give $P_n(t) = T_n(1+2t)/T_n(3)$ as the best polynomials to minimize μ_n, where

$$(1) \qquad T_n(x) = [(x + \sqrt{x^2-1})^n + (x - \sqrt{x^2-1})^n]/2\ ,$$

and

$$T_n(3) = (\alpha^n + \alpha^{-n})/2,\ \alpha = 3 + \sqrt{8} \approx 5.828\ .$$

The infinite matrix $A = (a_{nk})$, associated with these polynomials, has entries

$$(2) \qquad a_{nk} = \begin{cases} 1/T_n(3), & k = 0 \\ \dfrac{2^{2k-1}}{T_n^{(3)}}\left[2\binom{n+k}{n} - \binom{n+k-1}{n-k}\right], & 0 < k \leqq n \\ 0, & k > n\ . \end{cases}$$