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COMPACT OPERATORS OF THE FORM uC_φ

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If A is the disc algebra, the uniform algebra of functions analytic on the open unit disc D and continuous on its closure, and if $u, \varphi \in A$ with $\|\varphi\| \leqq 1$, then the operator uC_φ is defined on A by $uC_\varphi\colon f(z) \to u(z)f(\varphi(z))$. In this note we characterize compact operators of this form and determine their spectra.

We recall that a bounded linear operator T from a Banach space B_1 to a Banach space B_2 is *compact* if given a bounded sequence $\{x_n\}$ in B_1, there exists a subsequence $\{x_{n_k}\}$ such that $\{Tx_{n_k}\}$ converges in B_2.

If $\varphi\colon \bar{D} \to \bar{D}$, we let φ_n denote n^{th} the iterate of φ, i.e., $\varphi_0(z) = z$ and $\varphi_n(z) = \varphi(\varphi_{n-1}(z))$ for $z \in \bar{D}$ and $n \geqq 1$. Our main result is the following.

THEOREM. *Let $u \in A$, $\varphi \in A$, $\|\varphi\| \leqq 1$ and suppose φ is not a constant function.*

I. *The operator uC_φ is compact if, and only if, $|\varphi(z)| < 1$ whenever $u(z) \neq 0$.*

II. *Suppose uC_φ is compact and let $z_0 \in \bar{D}$ be the unique fixed point of φ for which $\varphi_n(z) \to z_0$ for all $z \in D$. If $|z_0| = 1$, then uC_φ is quasinilpotent, while if $|z_0| < 1$, the spectrum $\sigma(uC_\varphi) = \{u(z_0)\varphi'(z_0)^n \mid n$ is a positive integer$\} \cup \{0, u(z_0)\}$.*

1. Characterization of compact uC_φ.

We first consider the easy case in which φ is a constant function.

THEOREM 1.1. *Suppose $u \in A$ and $\varphi(z) = a \in \bar{D}$ for all $z \in \bar{D}$. Then uC_φ is compact.*

Proof. Since $\varphi(z) = a$ for all $z \in \bar{D}$, $(uC_\varphi)f(z) = u(z)f(\varphi(z)) = f(a)u(z)$. Therefore the range of uC_φ is one-dimensional and so uC_φ is compact.

We next give a necessary and sufficient condition that uC_φ be a compact operator for those φ which are not constant functions.

THEOREM 1.2. *Suppose $u \in A$, $\varphi \in A$, $\|\varphi\| \leqq 1$ and φ is not a constant function. Then uC_φ is a compact operator on A if, and only if, $|\varphi(z)| < 1$ whenever $u(z) \neq 0$.*