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WHEN IS A POINT BOREL?

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Let X be a topological space. We investigate the question: When is a point (of X) Borel? In relation to this, we establish the equivalence of (a) Each point (singleton) is Borel, (b) Each point is the intersection of closed set and a G_δ, (c) The derived set of each point is Borel, (d) The derived set of each point is an F_σ, (e) The derived set of each subset is Borel, and (f) The derived set of each subset is an F_σ. Conditions (a), (b), (c), and (d) are also equivalent for a fixed point. As a separation axiom (a) is shown to lie strictly between T_1 and T_0. A number of examples are given and the work of other authors discussed.

0. Introduction. Consideration of the question posed in the title for a particular case led to the development of the theorem below.

THEOREM 0.1. *The following are equivalent conditions for a topological space X.*

(a) *Each point (singleton) of X is a Borel set.*

(b) *Each point of X is the intersection of a closed set and a G_δ set.*

(c) *The derived set of each point of X is a Borel set.*

(d) *The derived set of each point of X is an F_σ set.*

(e) *The derived set of each subset of X is a Borel set.*

(f) *The derived set of each subset of X is an F_σ set.*

The initial discovery was the implication (a) $\Rightarrow$ (b). Using it, one can show directly[1] that the T_0 separation axiom is satisfied if each point is Borel, with the latter condition certain for T_1 spaces.

In [1], C. E. Aull and W. J. Thron introduce and study a number of separation axioms between T_0 and T_1, each of which is classified by some property of derived sets. In Theorem 3.1 of [1], they prove that $\{p\}'$ is closed (which is taken as a separation axiom, T_D) if and only if there is a closed set F and an open set U such that $\{p\} = F \cap U$, for all $p \in X$. With this as a catalyst, the equivalence of (b) and (d) is established and "each point is Borel" is fit into the classification scheme of Aull and Thron as follows:

[1] We shall do it differently.