PACIFIC JOURNAL OF MATHEMATICS
Vol. 80, No. 1, 1979

A RADON-NIKODYM THEOREM FOR *-ALGEBRAS

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A noncommutative Radon-Nikodym theorem is developed in the context of *-algebras. Previous results in this direction have assumed a dominance condition which results in a bounded "Radon-Nikodym derivative". The present result achieves complete generality by only assuming absolute continuity and in this case the "Radon-Nikodym derivative" may be unbounded. A Lebesgue decomposition theorem is established in the Banach *-algebra case.

1. Definitions and Examples. Although there is a considerable literature on noncommutative Radon-Nikodym theorems, all previous results have needed a dominance, normality or other restriction [1-4, 7, 8, 12, 15-18]. Moreover, most of these results are phrased in a von Neumann algebra context. In this paper, we will obtain a general theorem on a *-algebra with no additional assumptions.

Let $\mathscr{A}$ be a *-algebra with identity I. A **-representation* of $\mathscr{A}$ on a Hilbert space H is a map π from $\mathscr{A}$ to a set of linear operators defined on a common dense invariant domain $D(\pi) \subseteqq H$ which satisfies:

(a) $\pi(I) = I$;

(b) $\pi(AB)x = \pi(A)\pi(B)x$ for all $x \in D(\pi)$ and $A, B \in \mathscr{A}$;

(c) $\pi(\alpha A + \beta B)x = \alpha\pi(A)x + \beta\pi(B)x$ for all $x \in D(\pi)$, $\alpha, \beta \in C$ and $A, B \in \mathscr{A}$;

(d) $\pi(A^*) \subset \pi(A)^*$ for all $A \in \mathscr{A}$.

The *induced topology* on $D(\pi)$ is the weakest topology for which all the operations $\{\pi(A): A \in \mathscr{A}\}$ are continuous [13]. A *-representation π is *closed* if $D(\pi)$ is complete in the induced topology. A *-representation π is *strongly cyclic* if there exists a vector x_0 such that $\pi(\mathscr{A})x_0 = \{\pi(A)x_0: A \in \mathscr{A}\}$ is dense in $D(\pi)$ in the induced topology [13]. We then call x_0 a *strongly cyclic vector*. Denoting the set of bounded linear operators on H by $\mathscr{L}(H)$, the *commutant* $\pi(\mathscr{A})'$ of π is

$$\pi(\mathscr{A})' = \{T \in \mathscr{L}(H): \langle T\pi(A)x, y\rangle = \langle Tx, \pi(A^*)y\rangle A \in \mathscr{A}, x, y \in D(\pi)\} .$$

Let v and w be positive linear functionals on $\mathscr{A}$. A sequence $A_i \in \mathscr{A}$ is called a (v, w) *sequence* if

$$\lim_{i\to\infty} v(A_i^*A_i) = \lim_{i,j\to\infty} w[(A_i - A_j)^*(A_i - A_j)] = 0 .$$

We now generalize various forms and strengthened forms of the classical concept of absolute continuity.