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NONFACTORIZATION IN COMMUTATIVE, WEAKLY SELF-ADJOINT BANACH ALGEBRAS

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A Banach algebra A is said to have "(weak) factorization" if for each $f \in A$, there exist $g, h \in A$ (resp. $n \geqq 1$ and $g, {}_1h_1, \cdots, g_n, h_n \in A$) such that $f = gh$ ($f = \Sigma g_j h_j$). Cohen's factorization theorem says that if A has bounded approximate identity, then A has factorization. The converse is false in general. This paper investigates various implications of factorization and weak factorization for commutative algebras that are weakly self-adjoint. (Defined below; these algebras include self-adjoint algebras.) The main result is Theorem 1.3: If the weakly self-adjoint commutative Banach algebra A of functions on the locally compact space X has weak factorization, then there exists $K > 0$ such that, for all compact subsets E of X, there exists $f \in A$ such that $||f|| \leqq K$ and $f \geqq 1$ on E. Applications of 1.3 are given. In particular it is shown that a proper character Segal algebra on $L^1(G)$, (G a LCA group) cannot have weak factorization.

We say that a Banach algebra B of complex-valued continuous functions on a topological space is *weakly self-adjoint* if there exists $K_0 > 0$ such that for each $f \in B$

$$(0.1) \qquad |f|^2 \in B \quad \text{and} \quad |||f|^2||_B \leqq K_0 ||f||_B^2 .$$

Obviously (0.1) is satisfied if B is self-adjoint ($f \in B$ implies $\bar{f} \in B$), or if B is a Banach ideal of a self-adjoint Banach algebra A of complex-valued continuous functions. Recall that B is called a *Banach ideal* of a Banach algebra A if $||f||_B \geqq ||f||_A$ and $fg \in B$, with $||fg||_B \leqq ||f||_B ||g||_A$ for all $f \in B, g \in A$. Dense Banach ideals have been called A-Segal algebras by Burnham [1] and others. It is worth mentioning that a self-adjoint Banach algebra B has weak factorization if and only if each element of B can be written as a linear combination of elements of the form $|f|^2, f \in B$ (this follows easily from the identity $f\bar{g} = 1/4 \sum_{k=0}^{3} f^k |f + i^k g|^2$).

Our work was motivated by an attempt to give a converse to Cohen's theorem for Segal algebras on abelian groups (defined in §2), and thereby extend the results of Burnham [1], Leinert [7], Wang [15], Yap [17] and others. Since such a Segal algebra cannot have bounded approximate units, that would prove that a proper Segal algebra cannot have weak factorization. We cannot prove that in full generality, but we are able to improve earlier results substantially.