PACIFIC JOURNAL OF MATHEMATICS
Vol. 80, No. 2, 1979

ON COMMON FIXED POINT SETS OF COMMUTATIVE MAPPINGS

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Let C be a compact convex subset of a locally convex topological vector space X. Anzai and Ishikawa recently proved that if $T_1, \cdots, T_n$ is a finite commutative family of continuous affine self-mappings of C, then $F(\sum_{i=1}^n \lambda_i T_i) = \bigcap_{i=1}^n F(T_i)$ for every λ_i such that $0 < \lambda_i < 1$ and $\sum_{i=1}^n \lambda_i = 1$, where $F(T)$ denotes the fixed point set of T. It is natural to question whether the conclusion of their theorem is dependent on the topological properties of X, C and T_i—in this case, the linear topology, the compactness and the continuity. We shall see that this is not; the theorem can be formulated in an algebraic context.

Our theorem, when applied to Hausdorff topological vector spaces, yields a better version of Anzai-Ishikawa's theorem (see Corollary 2).

DEFINITION 1. A subset B of a real vector space is said to be (algebraically) bounded if $\bigcap_{\varepsilon>0} \varepsilon(C - C) = \{0\}$, where $C = C_0(B)$, the convex hull of B.

Every bounded convex subset of a Hausdorff topological vector space is algebraically bounded. Every bounded subset of a locally convex Hausdorff topological vector space is algebraically bounded.

THEOREM 1. *Let C be a convex subset of a real vector space X and $T_1, \cdots, T_n$ a finite commutative family of affine self-mappings of C. If the set $D = \{T_1^{m_1} T_2^{m_2} \cdots T_n^{m_n} x: 0 \leqq m_i < \infty, i = 1, \cdots, n\}$ is bounded for each $x \in C$, then $F(\sum_{i=1}^n \lambda_i T_i) = \bigcap_{i=1}^n F(T_i)$ for every $0 < \lambda_i < 1$ with $\sum_{i=1}^n \lambda_i = 1$.*

LEMMA 1. *Let x_n be a sequence in a Banach space such that $x_n \to x$. Then the sequence y_n defined by*

$$y_n = (1/2^n)(x_0 + {}_nC_1 x_1 + \cdots + {}_nC_i x_i + \cdots + x_n)$$

converges to x.

Proof. For an arbitrary $\varepsilon > 0$, choose m such that $\|x_i - x\| < \varepsilon/2$ for $i \geqq m$. Choose $N \geqq m$ such that

$$1/2^n(1 + {}_nC_1 + \cdots + {}_nC_{m-1}) < \varepsilon/(2D)$$

for all $n \geqq N$, where D is a number such that $\|x_i - x\| \leqq D$ for all $i \geqq 0$. Then