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ON CHARACTERIZATIONS OF EXPONENTIAL POLYNOMIALS

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This paper considers some characterizations of exponential polynomials in $C(G)$, the set of all continuous complex valued functions on a σ-compact locally compact Abelian group G. For $f \in C(G)$, U_f will denote the subspace of $C(G)$ obtained by taking finite linear combinations of translates of f. It is known that f is an exponential polynomial if and only if U_f is of finite dimension. Our main result is to show that f is an exponential polynomial when U_f is closed in $C(G)$ if $C(G)$ is given the topology of convergence uniform on all compact subsets of G.

Further characterizations of exponential polynomials are given when G is real Euclidean n-space, R^n.

A function $b \in C(G)$ is additive if $b(x+y) = b(x) + b(y)$ for all $x, y \in G$ and $g \in C(G)$ is an exponential if $g(x+y) = g(x)g(y)$ for all $x, y \in C(G)$. An exponential polynomial is a finite linear combination of terms $h = b_1^{q_1} b_2^{q_2} \cdots b_m^{q_m} g$ where $b_1, b_2, \cdots, b_m$ are additive, $q_1, q_2, \cdots, q_m$ are nonnegative integers and g is an exponential.

If f is an exponential polynomial, it is easy to see that U_f is finite dimensional. For if h is as above, then $T_\alpha h\colon x \rightarrow h(x-\alpha)$ is a finite linear combination of terms $b_1^{r_1} b_2^{r_2} \cdots b_m^{r_m} g$ for each $\alpha \in G$ where $r_j = 0, 1, \cdots, q_j$ for $j = 1, 2, \cdots, m$. A result of Engert [5] shows that if U_f is finite dimensional, then f is an exponential polynomial. The proof of this result when G is any σ-compact locally compact Abelian group is naturally more involved than when G is merely R or R^n. Proofs for the case of $C(R)$ may be found in Anselone and Korevaar [1] and Loewner [8] who also refers to $C(R^n)$.

Throughout this paper, the only topology considered on $C(G)$ is that of convergence uniform on all compact subsets of G. With G being σ-compact, let G be the countable union of compact sets K_p. Let $S_p(f) = \sup\{|f(x)|: x \in K_p\}$ and $d(f, g) = \sum_{p=1}^{\infty} 2^{-p} \min(1, S_p(f-g))$ for $f, g \in C(G)$. Then d is a metric for $C(G)$ and $C(G)$ is complete in this metric.

With such a topology for $C(G)$, if U_f is finite dimensional, it is closed. The converse to this is shown here (Theorem 3) so that in $C(G)$,

$$f \text{ is an exponential polynomial} \Longleftrightarrow U_f \text{ is finite dimensional}$$
$$\Longleftrightarrow U_f \text{ is closed in } C(G)\,.$$