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EMBEDDING PARTIAL IDEMPOTENT d-ARY QUASIGROUPS

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It is shown that every finite partial idempotent d-quasigroup is embedded in a finite idempotent d-quasigroup.

1. Introduction. Evans [3] has proved that every partial Latin square of order n can be embedded in a Latin square of order $2n$. Equivalently, every partial quasigroups of order n can be embedded in a quasigroup of order $2n$. The connection between Latin squares and quasigroups is explained in [2]. Lindner [5] has proved that every idempotent partial quasigroup of order n can be embedded in an idempotent quasigroup of order 2^n, while Hilton [4], using a different technique, reduced this order to $4n$. After Cruse [1] gave a multidimensional analogue of Evans' theorem, Lindner [6] succeeded in proving an embedding theorem for idempotent ternary quasigroups. In the present paper, denoting by $N(p)$ the minimal order of d-quasigroups in which the partial idempotent d-quasigroup (P, p) is embedded, we show that (P, p) is embedded in an idempotent d-quasigroup (Q, q), such that $|Q| \leqq 2N(p)$ if d is odd and $|Q| \leqq 3N(p)$ if d is even.

For $d = 3$ this is an improvement on Lindner's result, but when $d = 2$ our construction gives a higher upper bound than Hilton's. The reason for this is that Hilton's construction relies on the fact that a partial quasigroup can be embedded in a quasigroup with the order doubled. This is not known to be true when $d > 2$ and a direct generalization of Hilton's construction cannot be applied.

2. Notation and definitions. If A is a set and $x \in A^d$, then x_i denotes the ith component of $x = (x_1, x_2, \cdots, x_d)$. If $x \in A$, $\bar{x} \in A^d$ is defined as $\bar{x} = (x, x, \cdots, x)$. Similar notation applies to the functions $f: X \rightarrow Y^d$ and $g: X \rightarrow Y$. For every $x \in X$

$$f(x) = (f_1(x), f_2(x), \cdots, f_d(x))$$

and for every $x \in X^d$, $\bar{g}(x) = (g(x_1), g(x_2), \cdots, g(x_d))$. The function Δ_A: $A \rightarrow A^d$ is defined as $\Delta_A(x) = \bar{x}$ for all $x \in A$. The restriction of $f: S \rightarrow T$ to $A \subseteqq S$ is denoted by $f|A$. We may take exception when f is a d-ary operation, in which case $f|A$ will often be abbreviated by f. When no danger of ambiguity exists, we do not distinguish between $h: S \rightarrow T$ and $g: S \rightarrow U$ if $h(x) = g(x)$ for every $x \in S$. The symbol $[x, y]$ denotes the d-tuple