PACIFIC JOURNAL OF MATHEMATICS
Vol. 80, No. 2, 1979

COMMUTANTS AND THE OPERATOR EQUATION $AX=\lambda XA$

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Suppose A is a bounded operator on the Banach space $\mathscr{B}$ such that A or A^* is one-to-one. In this note, we point out a relation between the commutant of A, the commutants of its powers, and operators which intertwine A and λA, where λ is a root of unity. A consequence of this relation is that the commutants of A and A^n are different if and only if there is an operator Y, not zero, that satisfies $AY=\lambda YA$, where $\lambda^n=1$, $\lambda\neq 1$. Combining this with Rosenblum's theorem, we see that if the spectra of A and XA are disjoint, the commutant of A is the same as that of A^2. We also use the theorem to give a counterexample to a conjecture of Deddens concerning intertwining analytic Toeplitz operators.

If A, B, and X are bounded operators on $\mathscr{B}$, we say X *commutes* with A if $XA = AX$, and we say X *intertwines* A and B if $XA = BX$. The set of operators that commute with A, the commutant of A, will be denoted $\{A\}'$.

LEMMA. *Suppose A is an operator such that A or A^* is one-to-one, and λ is a primitive nth root of 1. If X commutes with A^n, the operators $Y_i = \sum_{j=0}^{n-1} \lambda^{ij}A^{n-j-1}XA^j$, for $i = 0, 1, \cdots, n-1$, are the unique operators such that $AY_i = Y_i(\lambda^i A)$ and $nA^{n-1}X = \sum_{i=0}^{n-1} Y_i$.*

Proof. Let $Y_i = \sum_{j=0}^{n-1} \lambda^{ij}A^{n-j-1}XA^j$.
Then

$$\begin{aligned} AY_i &= \sum_{j=0}^{n-1} \lambda^{ij}A^{n-j}XA^j = A^nX + \sum_{j=1}^{n-1} \lambda^{ij}A^{n-j}XA^j \\ &= XA^n + \sum_{j=1}^{n-1} \lambda^{ij}A^{n-j}XA^j \\ &= \sum_{k=0}^{n-1} \lambda^{i(k+1)}A^{n-k-1}XA^{k+1} \\ &= \left(\sum_{k=0}^{n-1} \lambda^{ik}A^{n-k-1}XA^k\right)(\lambda^i A) = Y_i(\lambda^i A) \,. \end{aligned}$$

Since $\sum_{i=0}^{n-1} \lambda^{ij} = 0$ when $j \neq 0$, and the sum is n when $j = 0$,

$$\begin{aligned} \sum_{i=0}^{n-1} Y_i &= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} \lambda^{ij}A^{n-j-1}XA^j \\ &= \sum_{j=0}^{n-1} A^{n-j-1}XA^j \sum_{i=0}^{n-1} \lambda^{ij} = nA^{n-1}X \,. \end{aligned}$$