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A SELECTION THEOREM FOR GROUP ACTIONS

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Let a Polish group G act continuously on a Polish space X, inducing an equivalence relation E. Let E_Y be the restriction of E to an invariant Borel subset Y of X. Assume E_Y is countably separated. Then it has a Borel transversal.

Throughout, let Γ be a continuous action of a Polish group G on a Polish space X. Thus X is a separable space admitting a complete metric, while G is a topological group whose topology is separable and admits a complete metric, and Γ is a continuous function $G \times X \rightarrow X$ satisfying $\Gamma(g^{-1}, \Gamma(g, x)) = x$ and $\Gamma(g, \Gamma(h, x)) = \Gamma(gh, x)$ for all $x \in X$ and $g, h \in G$. We write gx for $\Gamma(g, x)$, and for subsets of X write gA for $\{gx: x \in A\}$. Γ induces an equivalence relation E on X: xEy iff $gx = y$ for some $g \in G$. $W \subset X$ is *invariant* if $gW = W$ for all $g \in G$. Let $Y \subset X$ be an invariant Borel set, E_Y the restriction of E to Y. A *transversal* or *selector-set* for an equivalence relation is a set composed of exactly one representative from each equivalence class. Let us assume E_Y is *countably separated*, i.e., that there exist invariant Borel $Z_0, Z_1, Z_2, \cdots \subset Y$ such that for all $x, y \in Y$:

$$xEy \longleftrightarrow \forall m(x \in Z_m \longleftrightarrow y \in Z_m) \tag{0}$$

our goal is to prove the following selection result:

THEOREM. Under the above hypotheses, E_Y has a Borel transversal. It should be mentioned that a number of special cases and overlapping results have been known to and applied by C^*-algebraists for some time now. The construction of the required transversal proceeds in four stages.

Stage A. It will prove convenient to reserve the letters m, n plain and with subscripts to range over the set I of natural numbers, and to reserve s, t plain and with subscripts to range over the set Q of finite sequences of natural numbers. We let s^*m denote the *concatenation* of s and m, i.e., s with m tacked on at the end. We wish to define Borel sets $A(s)$ for overy $s \in Q$ of even length.

Case 1. $s =$ the empty sequence $\varnothing$. Set $A(\varnothing) = Y$.

Case 2. $s = a$ sequence (m, n) of length two. Set $A((m, n)) = Z_m$