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A CYCLIC INEQUALITY AND A RELATED EIGENVALUE PROBLEM

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A cyclic sum $S(\underline{x})=\Sigma x_i/(x_{i+1}+x_{i+2})$ is formed with the N components of a vector $\underline{x}$, where $x_{N+1}=x_1$, $x_{N+2}=x_2$, and *where all denominators are positive and all numerators non-negative*. It is known that the inequality $S(\underline{x})\geqq N/2$ does not hold for even $N\geqq 14$; this result is derived in a uniform manner by considering a related algebraic eigenvalue problem. Numerical evidence is presented for the conjecture that this cyclic inequality is true for even $N\leqq 12$ and odd $N\leqq 23$.

The corresponding cyclic inequality, namely the question for what value of N

$$S(\underline{x}) \geqq N/2$$

holds, has been investigated by many mathematicians (cf. Mitrinović [7] and the references given there). In §1 we prove in a unified manner that the inequality does not hold for even $N \geqq 14$. The method is based on the idea used first by Lighthill for $N = 20$ [4] and then by several other authors. The argument indicates why the case $N = 12$ remains still unresolved. Some properties of this type of solution are described in §2. Section 3 deals with numerical results that strongly suggest that the inequality is valid for $N = 12$ and, if N is odd, for $N = 23$. These numerical results definitely represent stationary values of the cyclic sum, and we are inclined to believe that they are indeed global minima. A connection between the inequality above and a related inequality with indices reversed is considered in the last section. In the Appendix some examples are listed for $N = 14, 25$ and 27.

1. *The linear cyclic inequality.* By considering the cyclic sum $S(\underline{x})$ it is obvious that for any N there exists a vector for which

$$S(\underline{x}) = N/2$$

holds, namely $x_i = 1$ for $i = 1, 2, \cdots, N$. If N is even, there exists also a wider class of "nominal" vectors,

$$x_i^0 = \begin{cases}(1+\alpha)/2 & \text{for } i \text{ odd}\\ (1-\alpha)/2 & \text{for } i \text{ even}\end{cases} \quad 0 \leqq \alpha \leqq 1\,, \tag{1.1}$$