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NONLINEAR DIFFERENTIAL EQUATIONS WITH MONOTONE SOLUTIONS

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The differential equation $dy^4/dt^4 - y = 0$ has as a fundamental set of solutions $\sin t, \cos t, e^t$, and e^{-t}. The latter of these is distinguished by the properties of being positive and strictly decreasing to zero as $t \to \infty$. As such, e^{-t} is the prototype of the "monotone solution" whose existence will be demonstrated for a large class of nonlinear differential equations of even order.

Our method will be restricted to differential equations of order $2n$ which can be written as second order systems of the form

$$x'' = f(x, t) \tag{1.1}$$

where $x \in R^n$ and f is a continuous function from $R^n \times [0, \infty)$ into R^n satisfying other conditions to be formulated in §2. Without resolving the question of what scalar equations allow such a representation, it is clear that our considerations will include nonselfadjoint linear fourth order equations (see [5]), equations of the form

$$(p_1 y'')'' = f_2(y, y'', t)$$
$$[p_2(p_1 y'')'']'' = f_3(y, y'', y^{iv}, t) ,$$

and similar equations of higher order.

In case (1.1) is linear and

$$x'' = A(t)x \tag{1.2}$$

where $A(t) = (a_{ij}(t))$ is a continuous $n \times n$ matrix, criteria for the existence of monotone solutions of (1.2) are well known. In particular, by letting $w = -'x$, (1.2) can be written as a first order system of the form

$$\begin{pmatrix} x \\ w \end{pmatrix}' = -\begin{pmatrix} 0 & I \\ A(t) & 0 \end{pmatrix}\begin{pmatrix} x \\ w \end{pmatrix} . \tag{1.3}$$

According to Hartman [2; Ch 14, Theorem 2.1], the condition $a_{ij}(t) \geqq 0$ for $1 \leqq i, j \leqq n$, and $0 \leqq t < \infty$ assures the existence of a nontrivial solution of (1.3) for which $x_i(t) \geqq 0$ and $x_i'(t) \leqq 0$ for $1 \leqq i \leqq n$ and $0 \leqq t < \infty$. Since $x(t)$ also satisfies (1.2), these results readily carry over to linear second order systems.

Nonlinear problems of the form (1.1) have also been studied by Hartman and Wintner [3] in terms of the related first order systems.