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AUTOMORPHISM GROUPS RETRACTING ONTO SYMMETRIC GROUPS

MATTHEW GOULD AND HELEN H. JAMES

The main result of this note is that if a group G retracts onto the symmetric group S_n (n finite) then G is isomorphic to the automorphism group of the nth direct power of a multi-unary algebra (equivalently: G is isomorphic to the automorphism group of an algebra that is free on a basis of n elements). It will be shown that the converse fails for all $n > 1$, but a restricted form of the converse will be proved.

It was noted by G. Birkhoff in [1] that for any group G the right translations of G are precisely the automorphisms of the algebra defined on G by taking the left translations as operations. Thus our main result stated above is true for $n = 1$, in which case the class of groups in question is simply the class of all groups. Moreover, as the algebra of left translations is easily seen to be freely generated by any one of its elements, the equivalent formulation of our result is also true for $n = 1$. We therefore stipulate that $n > 1$ throughout the sequel.

1. Preliminaries. Concepts and notations of universal algebra used here and not explicitly defined are taken from Grätzer [5], while group and semigroup teminology comes from Hall [6] and Clifford and Preston [2] respectively. Additionally, the notations Aut$(\mathfrak{A})$ and End$(\mathfrak{A})$ will denote respectively the automorphism group and endomorphism monoid of an algebra $\mathfrak{A}$, and the term *rigid* will be applied to an algebra $\mathfrak{A}$ satisfying $|\text{End}(\mathfrak{A})| = 1$. An algebra is said to be *multi-unary* if all its operations are unary.

We shall utilize the following four theorems from the literature. The first (but for a slight modification) and second come from the first author's work [3]; the first characterizes the endomorphism monoids of direct powers, while the second (with its obvious converse) characterizes the nontrivial automorphism groups of direct squares (thereby implying our main result in the case $n = 2$).

THEOREM 1.1. *Given a monoid M, the following are equivalent.*

(a) $M \cong \text{End}(\mathfrak{A}^n)$ *for some algebra* $\mathfrak{A}$.

(b) *There exist an n-ary operation* [ ] *on M and distinct elements $d_1, \cdots, d_n$ of M satisfying the identities*

(b.1) $d_i d_j = d_i$