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ON SELF-ADJOINT DERIVATION RANGES

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The properties of those operators on a Hilbert space which induce a derivation whose range after closure is self-adjoint are studied. Such operators are termed *D*-symmetric. A characterization of compact *D*-symmetric operators is given. Normal derivations are considered, and an example of an irreducible, not essentially normal, *D*-symmetric operator is presented.

Let $\mathscr{L}(\mathscr{H})$ denote the bounded linear operator on a Hilbert space $\mathscr{H}$. For $A \in \mathscr{L}(\mathscr{H})$ define a linear operator

$$\Delta_A: \mathscr{L}(\mathscr{H}) \longrightarrow \mathscr{L}(\mathscr{H})$$

as follows

$$\Delta_A: X \longrightarrow AX - XA$$

for all $X \in \mathscr{L}(\mathscr{H})$. Then Δ_A is an inner derivation on $\mathscr{L}(\mathscr{H})$ and remarkably enough all (linear) derivations on $\mathscr{L}(\mathscr{H})$ are of this form (see [11], [12] and [18]). The properties of inner derivations, their spectrum [13], norm [20] and ranges [2], [10], [21], [23] have been scrutinized carefully in recent years. In the paper we wish to consider the class of operators which have self-adjoint derivation ranges, at least after one closes in the norm topology.

DEFINITION. A operator $A \in \mathscr{L}(\mathscr{H})$ is D-symmetric if (range $\Delta_A)^- =$ (range $\Delta_{A^*})^-$ (the — indicates closure in the norm topology). We denote range Δ_A by $\mathscr{R}(\Delta_A)$. We denote the class of D-symmetric operotors by $\mathscr{S}$. Obviously A is D-symmetric if and only if $\mathscr{R}(\Delta_A)^-$ is a self-adjoint subspace of $\mathscr{L}(\mathscr{H})$. The concept of D-symmetric was introduced by Bunce and Williams.

Another paper [1] on this topic appeared at the same time as this one, and we have modified our terminology in accordance with theirs. On one occasion a more general result appears in [1] and in that instance (Theorem 3) we have merely stated our result, which is needed elsewhere, while omitting the proof.

The paper has been expanded to include an example of an irreducible D-symmetric operator which is not essentially normal.

1. General considerations.

We would like to explore the class $\mathscr{S}$ in this paper. We begin by proving a very simple yet often-times useful lemma concerning membership in $\mathscr{S}$.