PACIFIC JOURNAL OF MATHEMATICS
Vol. 82, No. 1, 1979

ON UNITARY AUTOMORPHISMS OF SOLVABLE LIE ALGEBRAS

OLDRICH KOWALSKI

Let V be a finite dimensional vector space over real numbers. An automorphism A of V is called unitary if it is semisimple and all its eigenvalues are complex units. Particularly, all periodic automorphisms, i.e., such that $T^k=$ identity for some integer k, are unitary. The aim of this paper is to prove the following Theorem. Let $\mathfrak{g}$ be an n-dimensional real Lie algebra admitting a unitary automorphism without nonzero fixed vectors. Then $\mathfrak{g}$ admits a periodic automorphism without nonzero fixed vectors and of order k, where $k \leq 5^{n/4}$ for n even, and $k \leqq 2.5^{(n-1)/4}$ for n odd.

The proof is based upon the detailed study of possible eigenvalues of admissible automorphisms of $\mathfrak{g}$. Yet our method is purely combinatorial—we do not make use of the Jacobi identities in $\mathfrak{g}$. Thus, the same method can be applied to nonassociative algebras or, more generally, to various tensor structures on vector spaces. As concerns applications to the differential geometry (generalized symmetric Riemannian spaces), see note at the end of this paper.

Comments:

(a) Obviously, an automorphism A of V is unitary if and only if V admits a scalar product $\langle\ ,\ \rangle$ such that $\langle Tu, Tv\rangle = \langle u, v\rangle$ for all $u, v \in V$.

(b) It is well-known ([1], [3]) that a finite dimensional Lie algebra admitting an automorphism without nonzero fixed vectors is solvable. Thus our theorem is essentially a result on solvable Lie algebras.

(c) For the validity of our theorem, it is not necessary to assume that the initial automorphism is semisimple. For, let A be an automorphism of $\mathfrak{g}$ and $A = S \cdot U$ the Jordan decomposition into the semisimple and the unipotent part. Then S is an automorphism of $\mathfrak{g}$ possessing the same eigenvalues as A. Particularly, if A is fixed-point free then so is S, and if all eigenvalues of A are complex units then S is unitary.

(d) The fact that $\mathfrak{g}$ admits a fixed-point free automorphism of finite order can be proved directly as follows (cf. also [2]): Let A be the given unitary, fixed-point free automorphism of $\mathfrak{g}$; then A can be represented by a diagonal matrix $(\theta_1, \cdots, \theta_n)$ belonging to