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ON A THEOREM OF MURASUGI

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1. Let $l=k_1\cup k_2$ be a 2-component link in S^3, with k_2 unknotted. The 2-fold cover of S^3 branched over k_2 is again S^3; let $k_1^{(2)}$ be the inverse image of k_1, and suppose that $k_1^{(2)}$ is connected. How are the signatures $\sigma(k_1)$, $\sigma(k_1^{(2)})$ of the knots k_1 and $k_1^{(2)}$ related? This question was considered (from a slightly different point of view) by Murasugi, who gave the following answer [Topology, 9 (1970), 283–298].

THEOREM 1 (*Murasugi*).

$$\sigma(k_1^{(2)})=\sigma(k_1)+\xi(l)\ .$$

Recall [4] that the invariant $\xi(l)$ is defined by first orienting l, giving, an oriented link $\bar{l}$, say, and then setting $\xi(l)=\sigma(\bar{l})+Lk(\bar{k}_1,\bar{k}_2)$, where σ denotes signature and Lk linking number.

In the present note we shall give an alternative, more conceptual, proof of Theorem 1, and in fact obtain it as a special case of a considerably more general result.

The idea of our proof is the following. If $l = l_1 \cup l_2$ is a link, partitioned into two sublinks l_1 and l_2, then the 2-fold branched covers over l_1, l_2, and the whole of l, are all quotients of a $\boldsymbol{Z}_2 \oplus \boldsymbol{Z}_2$-cover branched over l. After possibly multiplying by 2, the diagram consisting of these branched covers bounds a corresponding diagram of 4-manifolds, and the signatures of the various links involved are expressible in terms of the signatures of these 4-manifolds (and the euler numbers of the branch sets); see e.g., [3]. The result is then a consequence of a relation among these 4-manifold signatures (Lemma 1).

This more general setting requires that we consider links in 3-manifolds other than homology spheres; in § 2 we discuss the signature in this context. (It becomes necessary to prescribe a particular 2-fold branched cover. However, we sacrifice some generality inasmuch as we restrict ourselves to oriented, null-homologous links: it would otherwise be necessary to prescribe a framing of the link as well.) In § 3 we set up the diagram of covering spaces, and in § 4 derive the relation between the signatures of the manifolds therein. Section 5 contains some consequences of this, including the appropriate generalization of Theorem 1.

All manifolds of dimensions 3 and 4 are to be oriented; manifolds of dimensions 1 and 2 are oriented only when this is explicitly