PACIFIC JOURNAL OF MATHEMATICS
Vol. 82, No. 1, 1979

CONGRUENCE LATTICES OF ALGEBRAS OF FIXED SIMILARITY TYPE, I

RALPH FREESE, WILLIAM A. LAMPE AND WALTER TAYLOR

We prove that if V is any infinite-dimensional vector space over any uncountable field F, then the congruence lattice (=subspace lattice) of V cannot be represented as a congruence lattice (of any algebra) without using at least $|F|$ operations. This refutes a long-standing conjecture—that one binary operation would always suffice.

Our result implies that the natural representation of Sub V as a congruence lattice is the best one, i.e., uses the minimum number of operations. This result is easy to obtain—see § 1.

In the remainder of the paper we find further necessary conditions for the representability of an algebraic lattice L with $<\kappa$ operations. Necessary and sufficient conditions seem impossible at this stage of knowledge. Part II of this paper (by W. A. Lampe) gives some interesting sufficient conditions for representability of L with one binary operation, for instance: the unit element of L is compact.

The conjecture which we have refuted dates back at least to 1959 and the theorem of Grätzer and Schmidt ([6], [3], [14]) that every algebraic lattice can be represented as the congruence lattice Con A of some unary algebra A; the hope was to use say a single binary operation $+$ to code the unary operations $f(x)$ as $x + a$ (with $a \in A$ depending on f). (See e.g., [9], [8, p. 209].)

By some results of § 3 and Part II, every algebraic lattice can be embedded as a principal ideal in an algebraic lattice which is representable with one binary operation, and also in one requiring κ operations. This may partly explain why the conjecture resisted settlement for so long.

§ 1 contains the main result. §§ 2 and 3 contain refinements and variations on the ideas of § 1, and § 4 contains some open problems.

Some of these results were announced in [12], [13], and [18].

The authors thank B. Jónsson, A. Day, and E. Nelson for many helpful comments, and acknowledge support from the National Science Foundation and support (for W. A. Lampe) from the Institute for Advanced Study during 1974-75. The other two authors heartily thank William Lampe for so successfully initiating this investigation, and especially for his crucial Lemma 1, which made everything here possible.