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CONSTRUCTIVE VERSIONS OF TARSKI'S FIXED POINT THEOREMS

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Let F be a monotone operator on the complete lattice L into itself. Tarski's lattice theoretical fixed point theorem states that the set of fixed points of F is a nonempty complete lattice for the ordering of L. We give a constructive proof of this theorem showing that the set of fixed points of F is the image of L by a lower and an upper preclosure operator. These preclosure operators are the composition of lower and upper closure operators which are defined by means of limits of stationary transfinite iteration sequences for F. In the same way we give a constructive characterization of the set of common fixed points of a family of commuting operators. Finally we examine some consequences of additional semi-continuity hypotheses.

1. Introduction. Let $L(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ be a nonempty *complete lattice* with *partial ordering* $\sqsubseteq$, *least upper bound* $\sqcup$, *greatest lower bound* $\sqcap$. The *infimum* $\perp$ of L is $\sqcap L$, the *supremum* $\top$ of L is $\sqcup L$. (Birkhoff's standard reference book [3] provides the necessary background material.) Set inclusion, union and intersection are respectively denoted by $\subseteq$, $\cup$ and $\cap$.

Let F be a *monotone* operator on $L(\sqsubseteq, \perp, \top, \sqcup, \sqcap)$ into itself (i.e., $\forall X, Y \in L$, $\{X \sqsubseteq Y\} \Rightarrow \{F(X) \sqsubseteq F(Y)\}$).

The fundamental theorem of Tarski [19] states that the set $fp(F)$ of *fixed points* of F (i.e., $fp(F) = \{X \in L: X = F(X)\}$) is a nonempty complete lattice with ordering $\sqsubseteq$. The proof of this theorem is based on the definition of the least fixed point $lfp(F)$ of F by $lfp(F) = \sqcap\{X \in L: F(X) \sqsubseteq X\}$. The least upper bound of $S \subseteq fp(F)$ in $fp(F)$ is the least fixed point of the restriction of F to the complete lattice $\{X \in L: (\sqcup S) \sqsubseteq X\}$. An application of the duality principle completes the proof.

This definition is not constructive and many applications of Tarski's theorem (specially in computer science (Cousot [5]) and numerical analysis (Amann [2])) use the alternative characterization of $lfp(F)$ as $\sqcup\{F^i(\perp): i \in N\}$. This iteration scheme which originates from Kleene [10]'s first recursion theorem and which was used by Tarski [19] for complete morphisms, has the drawback to require the additional assumption that F is *semi-continuous* ($F(\sqcup S) = \sqcup F(S)$ for every *increasing* nonempty *chain* S, see e.g., Kolodner [11]).