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THE DIMENSION OF THE KERNEL OF A PLANAR SET

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Let S be a compact subset of R^2. We establish the following: For $1 \leqq k \leqq 2$, the dimension of $\ker S$ is at least k if and only if for some $\varepsilon > 0$, every $f(k)$ points of S see via S a common k-dimensional neighborhood having radius ε, where $f(1) = 4$ and $f(2) = 3$. The number $f(k)$ in the theorem is best possible.

We begin with some definitions: Let S be a subset of R^d. For points x and y in S, we say x *sees* y *via* S if the segment $[x, y]$ lies in S. The set S is *starshaped* if there is some point p in S such that, for every x in S, p sees x via S. The set of all such points p is called the (convex) *kernel* of S, denoted by $\ker S$.

A well-known theorem of Krasnosel'skii [5] states that if S is a compact set in R^d, then S is starshaped if and only if every $d+1$ points of S see a common point via S.

Although various results have been obtained concerning the dimension of the set $\ker S$ (Hare and Kenelly [3], Toranzos [6], Foland and Marr [2], Breen [1]), it still remains to set forth an appropriate analogue of the Krasnosel'skii theorem for sets whose kernel is at least k-dimensional, $1 \leqq k \leqq d$. Hence the purpose of this work is to investigate such an analogue for subsets of the plane.

The following terminology will be used. Throughout the paper, $\operatorname{conv} S$, $\operatorname{cl} S$, $\operatorname{int} S$, $\operatorname{bdry} S$, and $\ker S$ denote the convex hull, closure, interior, boundary, and kernel, respectively, of the set S. If S is convex, $\dim S$ represents the dimension of S. Finally for $x \neq y$, $R(x, y)$ denotes the ray emanating from x through y and $L(x, y)$ is the line determined by x and y.

2. The results.

We begin with the following theorem for sets whose kernel is 1-dimensional.

THEOREM 1. *Let S be a compact set in R^2. The dimension of $\ker S$ is at least 1 if and only if for some $\varepsilon > 0$, every 4 points of S see via S a common segment of radius ε. The number 4 is best possible.*

Proof. The necessity of the condition is obvious. Hence we need only establish its sufficiency.