PACIFIC JOURNAL OF MATHEMATICS
Vol. 82, No. 1, 1979

SUBSPACES OF POSITIVE DEFINITE INNER PRODUCT SPACES OF COUNTABLE DIMENSION

WERNER BÄNI

We deal with the following problem, proposed by I. Kaplansky in 1950: If V, $\bar{V}$ are subspaces of an inner product space (E, Φ) of countable dimension over any field k, when does there exist a metric automorphism of (E, Φ) mapping V onto $\bar{V}$? The present paper treats the case of *positive definite symmetric* spaces over $k = \boldsymbol{R}$. We shall characterize the orbits (under the orthogonal group of (E, Φ)) of a large class of subspaces V by two sequences of cardinals attached to V in a natural way (if e.g., $V^\perp = 0$ or $V = V^{\perp\perp}$ only a few of them are $\neq 0$; the case $V^\perp = 0$ is covered by work of H. Gross). However, classifying the subspaces not in this class is equivalent to classifying vector spaces F endowed with a sequence $\Omega_0, \Omega_1, \Omega_2, \cdots$ of positive definite forms.

1. Introduction.

1.1. Let E be a real vector space of countable dimension, equipped with a positive definite symmetric bilinear form $\Phi: E \times E \to \boldsymbol{R}$. We are concerned with the problem of classifying subspaces V of E with respect to metric automorphisms of (E, Φ) or, in other words, of describing the orbits of subspaces of E under the action of the orthogonal group of (E, Φ).

1.2. This problem of course originates from the wish to know how Witt's theorem generalizes to spaces of infinite dimension (the celebrated theorem says that every isometry between subspaces of a finite dimensional inner product space can be extended to the whole space). In fact, Kaplansky ([6], question 3, p. 16) stated the problem explicitly for arbitrary inner product spaces of countable dimension over any field k (in the case of uncountable dimensions the problem seems too nasty). Confirming a conjecture of Kaplansky, Gross [2] showed that in the presence of "sufficiently many" isotropic vectors (e.g., if Φ is alternate or if k is quadratically closed with char $(k) \neq 2$ and Φ is symmetric) the orbit of a subspace V may be characterized by seven cardinal numbers, namely the codimensions of neighboring spaces in the sublattice $\mathfrak{J}$ of $\mathfrak{L}(E)$ (the lattice of all linear subspaces of E) generated $\perp$-stably by V (i.e., $X \in \mathfrak{J} \Rightarrow X^\perp \in \mathfrak{J}$).

1.3. For positive definite forms over ordered fields - a case at the other extreme — the problem seems to be considerably harder.