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PERMUTATIONS OF THE POSITIVE INTEGERS WITH RESTRICTIONS ON THE SEQUENCE OF DIFFERENCES, II

PETER J. SLATER AND WILLIAM YSLAS VÉLEZ

In this paper we discuss the following conjecture:

Conjecture: **Let** $D = \{D_1, \cdots, D_n\}$, $D \subset N$, N **the set of positive integers. Then there exists a permutation of** N, **call it** $(a_k: k \in N)$ **such that** $\{|a_{k+1} - a_k|: k \in N\} = D$ **iff** $(D_1, \cdots, D_n) = 1$.

We also consider the following question:

Question: **For what sets** $D = \{D_1, \cdots, D_n\}$ **does there exist an integer** $M \in N$ **and a permutation** $\{b_k: k = 1, \cdots, M\}$ **of** $\{1, \cdots, M\}$ **such that** $\{|b_{k+1} - b_k|: k = 1, \cdots, M-1\} = D$.

We answer the conjecture and the following question in the affirmative if the set D **has the following property: For each** $D_r \in D$ **there is a** $D_s \in D$ **such that** $(D_r, D_s) = 1$.

In the following, we shall say that $(a_k: k \in N)$, N the set of positive integers, is a permutation if every integer $n \in N$ appears once and only once in the sequence $(a_k: k \in N)$. Set $d_k = |a_{k+1} - a_k|$.

In a previous paper, [1], we proved the following theorem.

THEOREM 1. *Let* $(m_j: j \in N)$ *be any sequence of positive integers. Then there exists a permutation* $(a_k: k \in N)$ *such that* $|\{i \mid d_i = j\}| = m_j$.

In constructing such permutations we could use infinitely many differences. We now ask if permutations of N can be constructed where the set of differences comes from a finite set. We make the following conjecture.

CONJECTURE. Let $D = \{D_1, \cdots, D_n\}$, $D \subset N$. Then there exists a permutation $(a_k: k \in N)$ such that $\{d_k: k \in N\} = D$ iff $(D_1, \cdots, D_n) = 1$, where $(D_1, \cdots, D_n)$ denotes the g.c.d. of the numbers, $D_1, \cdots, D_n$.

In this paper we show that the condition is necessary and that it is sufficient if corresponding to each $D_r \in D$, there is a D_s such that $(D_r, D_s) = 1$.

For $n = 1$, the condition that the g.c.d. be 1 gives that $D = \{1\}$. For the set, $D = \{1\}$, set $a_k = k$. Clearly, $\{d_k: k \in N\} = 0$.

LEMMA 1. *Let* $D = \{D_1, \cdots, D_n\}$, $D \subset N$. *If there exists a permutation* $(a_k: k \in N)$ *such that* $\{d_k: k \in N) = D$, *then* $(D_1, \cdots, D_n) = 1$.