PACIFIC JOURNAL OF MATHEMATICS
Vol. 82, No. 2, 1979

GENERAL PEXIDER EQUATIONS (PART II): AN APPLICATION OF THE THEORY OF WEBS

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Given open connected Ω, $\tilde{\Omega}\subseteq R^n$ and continuous T: $\Omega\rightarrow R$, F: $\tilde{\Omega}\rightarrow R$ both strictly monotonic in each variable separately. The equation $h\{T(x_1, \cdots, x_n)\}=F\{f_1(x_1), \cdots, f_n(x_n)\}$ for the unknowns h: $T(\Omega)\rightarrow R$ and π: $(f_1, \cdots, f_n)$: $\Omega\rightarrow\tilde{\Omega}$ can be interpreted within the theory of webs (the "Gewebe" of Blaschke-Bol). The web structure is then used to prove: any continuous solution π is uniquely determined on Ω by its value at two points of Ω; if a solution π is not continuous on Ω, then $\pi(\omega)$ is dense in $\tilde{\Omega}$ for every open ω in Ω; if a solution π is continuous at one point of Ω, it is continuous on Ω.

1. Formulation of results.

We consider the equation

$$h\{T(x_1, \cdots, x_n)\} = F\{f_1(x_1), \cdots, f_n(x_n)\}$$

for given T, F, with $h, f_1, \cdots, f_n$ the unknowns. Specifically, we assume given two sets $\Omega, \tilde{\Omega}$ in R^n and two functions $T: \Omega\rightarrow R$, F: $\tilde{\Omega}\rightarrow R$. Let Ω_i and $\tilde{\Omega}_i$ denote the projections of Ω and $\tilde{\Omega}$ onto the ith coordinate axis; by a *product mapping* $\pi: \Omega\rightarrow\tilde{\Omega}$ is understood the restriction to Ω of a mapping $(f_1, \cdots, f_n)$: $X_1^n\Omega_i\rightarrow R^n$ defined by the n functions $f_i: \Omega_i\rightarrow\tilde{\Omega}_i$. The above equation becomes $h\circ T = F\circ\pi$ with h: $T(\Omega)\rightarrow R$ and $\pi: \Omega\rightarrow\tilde{\Omega}$ the unknowns.

The present note is self-contained in that Part I [2] served only to indicate that the following hypotheses on T and F are not as restrictive as one might at first suppose. For the moment we assume only that:

(A.1) *T is continuous and strictly monotonic in each variable on Ω,*

(A.2) *F is strictly monotonic in each variable on $\tilde{\Omega}$,*

(A.3) *Ω is open and connected.*

THEOREM 1. *With* (A. 1, 2, 3) *assume that $h\circ T = F\circ\pi$ and $\bar{h}\circ T = F\circ\bar{\pi}$ hold for two product mappiugs $\pi, \bar{\pi}$ on Ω. If π and $\bar{\pi}$ are equal at the end points of some line segment $l\subset\Omega$ parallel to a coordinate axis, then π and $\bar{\pi}$ coincide on a set dense in Ω. Hence two continuous solutions must be identical on Ω if they agree at the end points of such a line segment.*

In the event $\Omega = R^n$ as in the classical case $T(x_1, \cdots, x_n) = x_1+$