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GENERAL PEXIDER EQUATIONS (PART I): EXISTENCE OF INJECTIVE SOLUTIONS

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Given open connected Ω, $\tilde{\Omega} \subseteq \boldsymbol{R}^n$ and given $T: \Omega \to \boldsymbol{R}$ continuous, $F: \tilde{\Omega} \to \boldsymbol{R}$ strictly monotonic, in each variable separately. The equation is $h \circ T = F \circ \pi$ for the unknowns $h: T(\Omega) \to \boldsymbol{R}$, $\pi: \Omega \to \tilde{\Omega}$ with $\pi = (f_1, \cdots, f_n)$ a product mapping - e.g., $h\{T(x, y)\} = F\{f(x), g(y)\}$. If T is one-one in each variable, then any continuous solution π must be injective or constant on Ω; conversely, if an injective solution π exists then T must be one-one in each variable separately.

1. Introduction. Given a subset $\Omega \subseteqq \boldsymbol{R}^n$ for $n \geqq 2$, let Ω_i denote its projection on the ith coordinate axis. By a *product mapping* $\pi: \Omega \to \tilde{\Omega} \subset \boldsymbol{R}^n$ is understood the restriction to Ω of a map $(f_1, \cdots, f_n): X_1^n \Omega_i \to \boldsymbol{R}^n$ defined by n functions $f_i: \Omega_i \to \tilde{\Omega}_i \subseteqq \boldsymbol{R}$. For given $T: \Omega \to \boldsymbol{R}$ and $F: \tilde{\Omega} \to \boldsymbol{R}$, equations of the form

$$h\{T(x_1, \cdots, x_n)\} = F\{f_1(x_1), \cdots, f_n(x_n)\} \tag{1}$$

for the unknowns $h: T(\Omega) \to \boldsymbol{R}$ and $\pi: \Omega \to \tilde{\Omega}$ are generalizations of Pexider equations[1]. For the most part the literature concerns the case in which T and F are specified, usually the sum and/or product of the arguments. In [3] C. T. Ng recently gave a uniqueness theorem for continuous solutions π, assuming T continuous but with $F(u_1, \cdots, u_n) = u_1 + \cdots + u_n$; a generalization to certain topological spaces appears in Ng [4] and [2]. A simple case of (1) was used by J. Lester and the author [5] to characterize Lorentz transformations in $\boldsymbol{R}^n$.

2. Formulation of results. Given $\Omega, \tilde{\Omega} \subseteqq \boldsymbol{R}^n$ for $n \geqq 2$ and given $T: \Omega \to \boldsymbol{R}$, $F: \tilde{\Omega} \to \boldsymbol{R}$. Henceforth assume:

(A-1) *T continuous in each variable separately,*
(A-2) *F one-to-one in each variable separately,*
(A-3) *Ω open and connected.*

THEOREM 1. *With* (A-1, 2, 3) *assume $T \circ h = F \circ \pi$ satisfied on Ω, where $h: T(\Omega) \to \boldsymbol{R}$ and where $\pi: \Omega \to \tilde{\Omega}$ is an injective product mapping. Then T must be strictly monotonic in each variable separately on Ω.*

The existence of an injective solution π then places a severe

[1] For literature see [1]; J. V. Pexider studied $h(x \dot{+} y) = f(x) \dot{+} g(y)$.