PACIFIC JOURNAL OF MATHEMATICS
Vol. 82, No. 2, 1979

A CONVERSE TO (MILNOR-KERVAIRE THEOREM) $\times R$ ETC...

MICHAEL H. FREEDMAN

One of the most puzzling questions in low dimensional topology is which elements $\alpha \in \pi_2(M)$, where M is a smooth compact 4-manifold, may be represented by a smoothly imbedded 2-sphere. This paper treats a stable version of the problem: When is there a smooth proper imbedding, h: $S^2 \times \boldsymbol{R} \hookrightarrow M \times \boldsymbol{R}$ by which the ends of $S^2 \times R$ are mapped to the ends of $M \times R$, and for which the composition

$$S^2 \xrightarrow{x \to (x,0)} S^2 \times \boldsymbol{R} \xrightarrow{h} M \times \boldsymbol{R} \xrightarrow{\pi} M$$

represents α?

If there is an h as above, we say that α is stably represented. We are able to determine precisely which α are stably represented when M is simply connected. In general, the vanishing of a finiteness obstruction $(\in K_0(Z[\pi_1(M)]))$ yields Poincare' imbeddings of S^2 in M. In the nonsimply connected case, sufficient information is obtained to carry out surgery $\times \boldsymbol{R}$, yielding an alternative construction of manifold structures on (4-dimensional Poincare' spaces) $\times \boldsymbol{R}$. All terminology will be smooth.

We say a class $\alpha \in \pi_2(M)$ is characteristic if the composition,

$$\pi_2(M) \xrightarrow{\text{Hur}} H_2(M; Z) \xrightarrow{(2)} H_2(M; Z_2) \xrightarrow{\partial} H_2(M, \partial; Z_2) \xrightarrow{P.D.^{-1}} H^2(M; Z_2) ;$$

carries α to $w_2(\tau(M))$. Otherwise, we say α is ordinary.

If α is characteristic and $N(\alpha) = 0$ there is a well defined number, $Arf(q(\alpha)) = 0$ or 1, which is the Arf invariant of a certain Z_2-quadratic form. When M is closed this Arf invariant is related to more familiar invariants by the formula $\text{Arf}(q(\alpha)) = \text{Hur}\,\alpha \cdot \text{Hur}\,\alpha$—signature $(M^4)/8 \pmod 2$. See [2] for details.

We say α has a spherical dual if there is a $\beta \in \pi_2(M)$ with $\lambda(\alpha, \beta) = 1$, where λ is the Wall-intersection form taking values in $Z[\pi_1(M)]$, see [5].

MAIN THEOREM (case: $\pi_1(M) = 0$). *α is stably represented if and only if α is ordinary or α is characteristic and* $\text{Arf}(q(\alpha)) = 0$.

MAIN THEOREM (case: $\pi_1(M) \neq 0$). *If α is stably represented, the Wall self intersection form $\mu(\alpha')$ is 0 for some immersion α' homotopic to α and if α is characteristic* $\text{Arf}(q(\alpha)) = 0$. *Conversely*