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WEAK LOCAL SUPPORTABILITY AND APPLICATIONS TO APPROXIMATION

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Perturbed optimization problems are studied using a weaker notion of local supportability than that developed by Ekeland and Lebourg. This weakening allows for a more comprehensive treatment of such problems. In particular we prove that nearest points exist densely for closed relatively weakly compact sets in spaces with locally uniformly convex norms and provide a simplified proof in this framework that a normed space with a Fréchet norm is an Asplund space.

This paper introduces a notion of a local subgradient for a lower semicontinuous function on a Banach space. This subgradient is required to satisfy a uniformity condition on a given bounded set in the space. The first section establishes the existence of such subgradients in weakly compactly generated Banach spaces. The following sections consist of applications of this result. Section two discusses generic differentiability of convex functions and contains a simple unified proof that spaces with Fréchet norms are Asplund spaces and weakly compactly generated spaces are weak Asplund spaces. In section three general perturbed optimization problems and the existence of farthest points are discussed. Section four shows, essentially, that any relatively weakly compact set C in a Banach space with a locally uniformly convex norm possesses a generic set of points with nearest points in C. This extends a known result for reflexive spaces.

Recently Ekeland and Lebourg [11] have introduced the notion of a local ε-support for a function and have profitably applied this to the study of perturbed optimization problems (including nearest and farthest points) and generic Fréchet differentiability. Rainwater [16] has provided a self-contained proof of this last result for convex functions which was deduced in [11] from more general perturbational theorems. Subsequently Lau [13] has applied the ε-supports to establish the existence of dense nearest points for any closed bounded set in any locally uniformly convex reflexive space. Since the results on approximation are deduced from the existence of appropriate Fréchet derivatives they are unavailable for application in more general spaces and hence do not provide best possible results (for nearest and farthest points, particularly). In this paper we introduce a more general notion of local support, examine the impli-