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A NOTE ON COMPACT OPERATORS WHICH ATTAIN THEIR NORM

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For Banach spaces X having the unit cell of $X^{**}w^*$-sequentially compact, the compact operators from X into a Banach space Y attain their norm in X^{**}. The same holds for weakly compact operators if, in addition, X has the strict Dunford-Pettis property. For Banach spaces X such that the quotient space X^{**}/X is separable and Y the space of absolutely summable sequences, a proper subset P_σ of the finite rank operators from X into Y is exhibited. The set P_σ is shown to consist of operators which attain their norm and to be norm-dense in the operator space.

Throughout, X and Y will be Banach spaces and $\mathscr{L}(X, Y)$ the space of bounded linear operators from X into Y. An operator $T \in \mathscr{L}(X, Y)$ attains its norm on the unit cell $S_{X^{**}}$ of X^{**} if $\|T\| = \|T^{**}x^{**}\|$ for some $x^{**} \in X^{**}$ of norm one. For general results on norm attaining operators and their density in $\mathscr{L}(X, Y)$, see [2]. A space X is said to have the strict Dunford-Pettis property [4 p. 137] if for all Banach spaces Y an arbitrary weakly compact operator $T \in \mathscr{L}(X, Y)$ maps weakly Cauchy sequences to strongly Cauchy sequences.

THEOREM 1. *Let X be a Banach space with $S_{X^{**}}$ sequentially compact in the $\sigma(X^{**}, X^*)$ topology. Then*

(i) *if $T \in \mathscr{L}(X, Y)$ is compact, T attains its norm on $S_{X^{**}}$. Thus, every compact operator with reflexive domain X attains its norm on S_X.*

(ii) *if $T \in \mathscr{L}(X, Y)$ is weakly compact and X has the strict Dunford-Pettis property, T attains its norm on $S_{X^{**}}$. In addition, therefore, if Y is reflexive, all operators attain their norms on $S_{X^{**}}$.*

Proof. There is a sequence $\{x_n\}$ in S_X satisfying $\|T\| < \|Tx_n\| + 1/n$. Let J_x be the canonical embedding of X into X^{**}. Since $\{J_x x_n\} \subseteqq S_{X^{**}}$ there exists a subsequence $\{x_{n_j}\}$ and an $x^{**} \in S_{X^{**}}$ such that $J_x x_{n_j} \xrightarrow{j} x^{**}$ in the $\sigma(X^{**}, X^*)$-topology. The sequence $\{x_{n_j}\}$ is weakly Cauchy in X, whence under either hypothesis there exists a subsequence $\{w_j\}$ of $\{x_{n_j}\}$ such that $\{Tw_j\}$ is norm-convergent to some $y \in Y$. Since $\{J_x w_j\}$ is $\sigma(X^{**}, X^*)$-convergent to x^{**} and $\{Tw_j\}$ is weakly convergent to y, we have $T^{**}x^{**} = J_y y$. Thus,