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THE DESCENDING CHAIN CONDITION RELATIVE TO A TORSION THEORY

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A well-known theorem of Hopkins and Levitzki states that any left artinian ring with identity element is left noetherian. The main theorem of this paper generalizes this to the situation of a hereditary torsion theory with associated idempotent kernel functor σ. It is shown that if a ring R with identity element has the descending chain condition on σ-closed left ideals, then R has the ascending chain condition on σ-closed left ideals.

The remainder of the paper generalizes some results of Faith and Walker concerning artinian and quasi-Frobenius rings. In the case that the localization functor $\mathscr{L}_\sigma$ is exact the following are obtained: (1) a sufficient condition for the ring R to have the descending chain condition on σ-closed left ideals and (2) characterizations of the condition that every σ-torsion-free injective left R-module is codivisible (projective).

In this paper R always denotes ring with identity element, and unless denoted to the contrary, all modules are members of the category R-mod of unital left R-modules.

A subfunctor σ of the identity functor on R-mod is called a *left exact radical* (or *idempotent kernel functor*) if σ is left exact and $\sigma(M/\sigma(M))=0$ for every module M. Such a σ naturally determines a torsion class $\mathscr{T}_\sigma = \{M \mid \sigma(M) = M\}$ and a torsion-free class $\mathscr{F}_\sigma = \{M \mid \sigma(M) = 0\}$. The pair $(\mathscr{T}_\sigma, \mathscr{F}_\sigma)$ forms a hereditary torsion theory in the sense of [2], [10], [13], [14] and [15]. Then $\mathscr{T}_\sigma$ is closed under submodules, homomorphic images, direct sums, and extensions of one member of $\mathscr{T}_\sigma$ by another; and $\mathscr{F}_\sigma$ is closed under submodules, direct products, injective hulls, and extensions of one member of $\mathscr{F}_\sigma$ by another. Also associated with σ is the *localization functor* $\mathscr{L}_\sigma$ as defined in [2], [4], [13] or [14]. The module $\mathscr{L}_\sigma(R)$ can be made into ring by defining multiplication in a natural way; this ring will be denoted by Q_σ. A torsion theory is called *perfect* in [2], [12] and [13] if $\mathscr{L}_\sigma(M) \cong Q_\sigma \otimes_R M$ for every module M. For additional details on the concepts discussed in this paragraph, the reader is referred to [2], [4], [9], [10], [13], [14], and their references.

A submodule N of M is called *σ-closed* if $M/N \in \mathscr{F}_\sigma$. The lattice of σ-closed submodules has been studied in [3], [5], [9], [12], [14], and [15]. Particular attention is usually given to chain conditions on σ-closed modules. We continue this investigation and focus our