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SETS OF INTEGERS CLOSED UNDER AFFINE OPERATORS-THE FINITE BASIS THEOREMS

D. G. HOFFMAN AND D. A. KLARNER

This paper is a continuation of investigations of sets T of integers closed under operations f of the form $f(x_1, \cdots, x_r) = m_1x_1 + \cdots + m_rx_r + c$, where $r, m_1, \cdots, m_r, c$ are integers satisfying $r \geqq 2$, $0 \notin \{m_1, \cdots, m_r\}$, and $\gcd(m_1, \cdots, m_r) = 1$. We have two goals here:

(1) to prove that $T = \langle f \mid A\rangle$ for some finite set A, where $\langle f \mid A\rangle$ denotes the "smallest" set containing A and closed under f, and

(2) to show that unless $|T| = 1$, T is a finite union of infinite arithmetic progressions, either all bounded below, or all bounded above, or all doubly infinite.

We shall lean heavily on the notation, definitions, and results of [1].

DEFINITION 1. Let $r \in \boldsymbol{P}$. An *r-ary affine* operator f on $\boldsymbol{Z}$ is an operator of the form

$$f(x_1, \cdots, x_r) = m_1x_1 + \cdots + m_rx_r + c\ ,$$

where $m_1, \cdots, m_r \in \boldsymbol{Z}\backslash\{0\}$, and $c \in \boldsymbol{Z}$. Let $\sigma(f) = m_1 + \cdots + m_r$, let $\rho(f) = r$.

We call f a *positive* operator if each $m_i \in \boldsymbol{P}$, a *prime* operator if $r \geqq 2$ and $\gcd(m_1, \cdots m_r) = 1$, and a *linear* operator if $c = 0$. Denote by $\mathscr{P}$ the set of all positive, prime, linear operators, and by $\mathscr{H}$ the set of all prime linear operators that are not positive. For each $f \in \mathscr{P}$, $\langle f + 1 \mid 0\rangle$ is a periodic set by Theorem 12 of [1]; let $\delta(f)$ be its smallest eventual period.

LEMMA 1. *Let $f \in \mathscr{P}$, let $a, s, t \in \boldsymbol{Z}$, with $(\sigma(f)-1)a + s \in \boldsymbol{N}$, and $(\sigma(f)-1)a + t \in \boldsymbol{P}$. Then $T = \langle f + \{s, t\} \mid a\rangle$ has an eventual period $\delta(f)\gcd(t - s, (\sigma(f)-1)a + t) = \delta(f)\gcd((\sigma(f)-1)a + s, (\sigma(f)-1)a + t)$.*

Proof. Define a sequence $(T_n \mid n \in \boldsymbol{P})$ of subsets of $\boldsymbol{Z}$ as follows: let $T_1 = \langle f + t \mid a\rangle$, and for $k \in \boldsymbol{P}$, let $T_{2k} = \langle f + s \mid T_{2k-1}\rangle$ and $T_{2k+1} = \langle f + t \mid T_{2k}\rangle$. Then certainly each T_n has an eventual period $\delta(f)((\sigma(f)-1)a + t)$, and further $T = \bigcup_{n \in P} T_n$. Thus T has an eventual period $\delta(f)((\sigma(f)-1)a + t)$. If $(\sigma(f)-1)a + s = 0$, we are done. Otherwise, we may interchange the roles of s and t in the argument above to conclude that T also has an eventual period of $\delta(f)((\sigma(f)-1)a + s)$.